Bounded-delay enumeration of regular languages

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Who

Joint work with Antoine Amarilli

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Outline

Introduction

Gray code for n-bit words

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$$w_1, w_2, \ldots, w_{2^n}$$

of $(a+b)^n$ such that w_i, w_{i+1} differ by exactly one bit.

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Other distances: definitions

We extend these definitions to other distances:

- the push-pop distance. Defined like the Levenshtein distance, but the basic operations are:
  - popL and popR, to delete the last (resp., the first) letter of the word; and
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- the push-pop-right distance. Defined like the push-pop distance, but only allow popR and $\mathrm{pushR}(\alpha)$ for $\alpha \in \Sigma$.

Other distances: first observations

languages orderable for push-pop-right $\subseteq$ languages orderable for push-pop $\subseteq$ languages orderable for Levenshtein.

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- Can we recognize them? (e.g., given a DFA)
- Can we always partition a regular language into a finite number of orderable languages? (as in $a^* + b^*$)
- When L is orderable, can we design an enumeration algorithm for it? With what delay? (poly, constant?)

Main results

Main results (Levenshtein and push-pop)

Let L be regular. We show:

- There exists $t \in \mathbb{N}$ and regular languages $L_1, \ldots, L_t$ such that

$$L = L_1 \sqcup \ldots \sqcup L_t$$

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  - → This shows L is orderable for Levenshtein iff it is for push-pop!
- When L is orderable for push-pop then, in a suitable pointer machine model, we have an algorithm that outputs push-pop edit scripts to enumerate L, with constant delay (i.e., independent from the current word length)

Enumeration algorithms with push-pop edit scripts

Let L regular, e.g., $(\epsilon + a)b^*$. GOAL: enumerate L with a delay that is independent from the length of the current word.

Enumeration algorithms with push-pop edit scripts

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Let L regular, e.g., $(\epsilon + a)b^*$. GOAL: enumerate L (in a certain sense) with a delay that is independent from the length of the current word. Example of a push-pop program for $(\epsilon + a)b^*$:

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int main{
  output();
  while (true) {
    pushR(b); output();
    pushL(a); output();
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The current word w_i is maintained on a (doubly-ended) queue (BLACKBOARD)

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An edit script is a sequence of push or pop operations that are executed between any two `output()` instructions. This push-pop program enumerates $(\epsilon + a)b^*$ with constant delay.

Proof of the lower bound

Lower bound

Theorem

For a regular language L, there exists $L_1, \ldots, L_t$ regular such that

$$L = L_1 \sqcup \ldots \sqcup \mathrm{L}_t$$

and each L_i is orderable for the push-pop distance. Moreover L cannot be partitioned into less than t orderable languages for the Levenshtein distance.

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We will now define this number t and show that it is optimal

Connectivity and compatibility of loopable states

Let $A = (Q, \Sigma, q_0, F, \delta)$ be a DFA for L. For $q \in Q$, define A_q to be A where the initial state and final state is q.

Definition: loopable state

A state $q \in Q$ is loopable if $\mathrm{L}(A_q) \neq \{\epsilon\}$. In other words, when there is a non-empty run that starts and ends at q.

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Definition: compatibility

Two loopable states $q, q' \in Q$ are compatible when $\mathrm{L}(A_q) \cap \mathrm{L}(A_{q'}) \neq \{\epsilon\}$.

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Note: The connectivity and compatibility relations of loopable states are reflexive but not transitive

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Interchangeability is the equivalence relation on loopable states that is defined to be the transitive closure of the union of the connectivity and compatibility relations. In other words, two loopable states $q, q' \in Q$ are interchangeable if there is a sequence $q = q_0, \dots, q_n = q'$ of loopable states such that for all $0 \leq i < n$, the states q_i and q_{i+1} are either connected or compatible.

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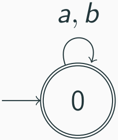
We then define t to be the number of interchangeable classes

Some examples follow

Example: $(a + b)^*$



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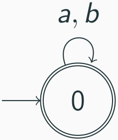

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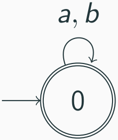

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$\implies$ $t = 1$

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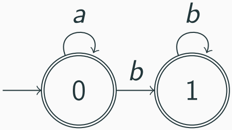

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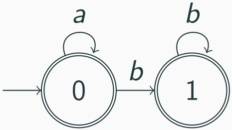

- Loopable states: 0 and 1

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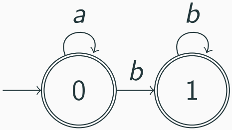

- Loopable states: 0 and 1
- 0 and 1 are connected, hence interchangeable

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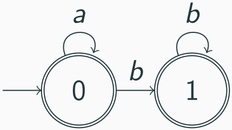

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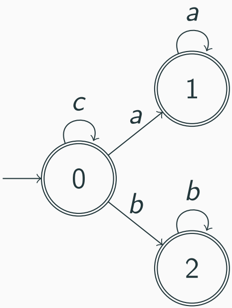
Example: $c^* a^* + c^* b^*$



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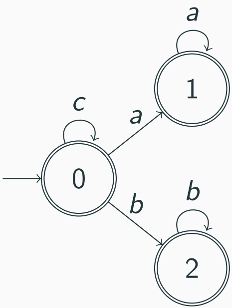

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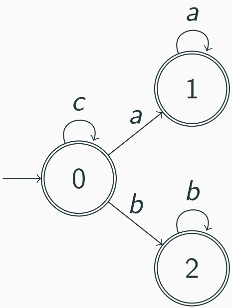

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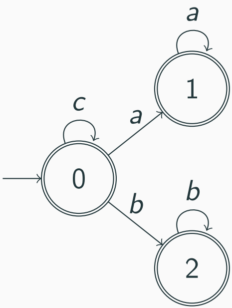

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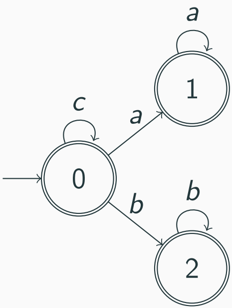

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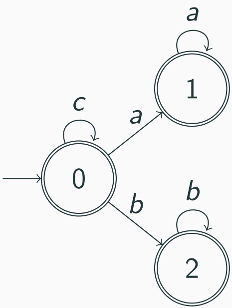

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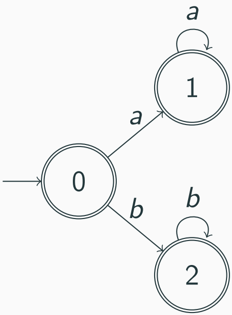

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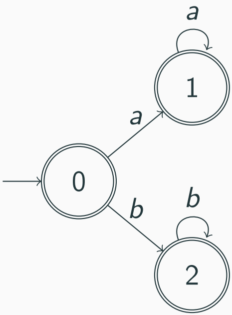

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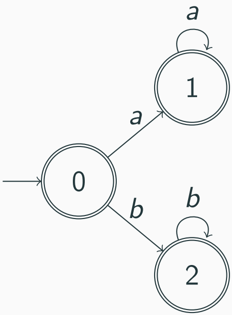

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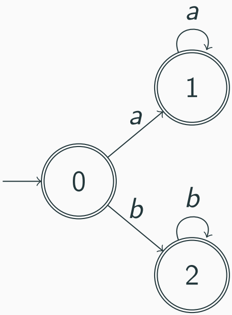

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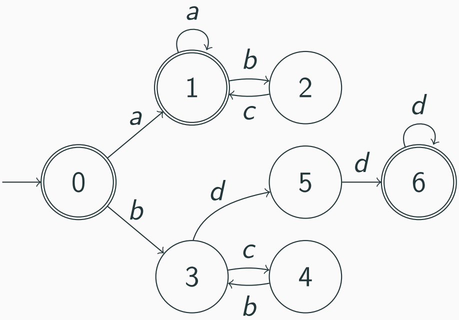
$\Longrightarrow$ $t = 2$

Example: $a(a+bc)^* + b(cb)^*ddd^*$



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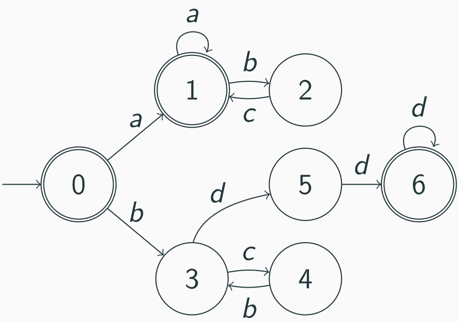

- Loopable states: $1, 2, 3, 4$ and 6

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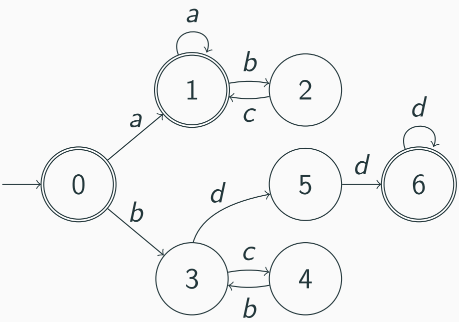

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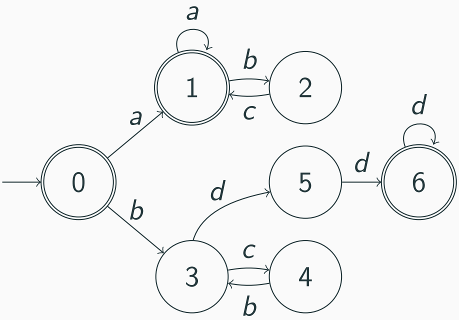

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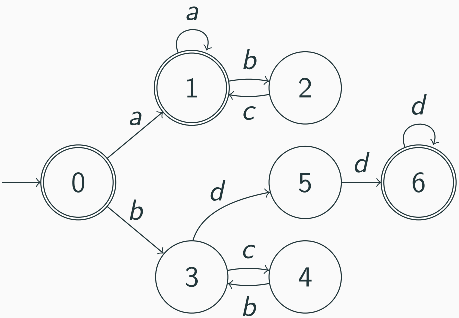

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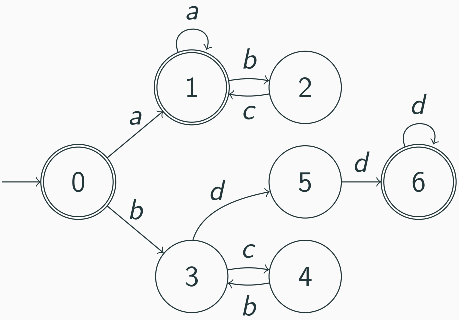

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$\Longrightarrow \quad t = 1$

The partition

Let $\mathcal{C}_1, \ldots, \mathcal{C}_t$ be the interchangeability classes of loopable states of A.

Definition

For $1 \leq i \leq t$, define

$$L_i = \{w \in \mathrm{L}(A) \mid \text{the run of } w \text{ goes through a state of } \mathcal{C}_i\}.$$

Also define

$$\mathrm{NL} = \{w \in \mathrm{L}(A) \mid \text{the run of } w \text{ does not use loopable states}\}.$$

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Proposition

We have $L = \mathrm{NL} \sqcup L_1 \sqcup \ldots \sqcup L_t$

Proof: (BLACKBOARD)

Proof of the lower bound

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Proposition

L cannot be partitioned into less than t languages that each are orderable for the Levenshtein distance.

Proof: we only do the case $t = 2$ and $\mathrm{NL} = \varnothing$ (so $L = L_1 \sqcup L_2$).
We prove (BLACKBOARD): for any distance $d \in \mathbb{N}$, there is a threshold $l \in \mathbb{N}$ such that for any two words $u \in L_1$ and $v \in L_2$ with $i \neq j$ and $|u| \geq l$ and $|v| \geq l$, we have $\delta_{\mathrm{Lev}}(u, v) > d$.
Indeed this is enough, using the same argument as for $a^* + b^*$

Proof (sketch) of the upper bound

Upper bound: existence of an ordering

We have shown:

Theorem

Given a DFA A, we can partition $\mathrm{L}(A)$ into

$$L = L_1 \sqcup \ldots \sqcup L_t$$

such that L cannot be partitioned into less than t orderable languages for the Levenshtein distance.

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We now show that each L_i is orderable for the push-pop distance

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Upper bound: existence

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Let δ_{pp} denote the push-pop distance on Σ^*

d-connectivity

Definition

Two words w, w' in a language L are d-connected in L if there exists a sequence $w_0, \ldots, w_n$ of words of L with $w_0 = w$, $w_n = w'$, and $\delta_{\mathrm{pp}}(w_i, w_{i+1}) \leq d$ for all $0 \leq i < n$.
We say that L is d-connected if every pair of words of L is d-connected in L

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- We show a kind of converse for finite languages in the next slide

d-connectivity implies $3d$-orderability for finite languages

Proposition

If L is finite and d-connected then it is $3d$-orderable.

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Proof: take a spanning tree T of $G_{L,d}$. For $n \in T$, let $h(n)$ be its depth. Apply the following algorithm to the root of T:

```
void visit(node n){
  if(h(n) is even){
    enumerate(n);
    for (child ch of n)
        visit(ch);
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  if(h(n) is odd){
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Example (BLACKBOARD)

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Example (BLACKBOARD)

Two consecutive nodes enumerated by this algorithm are at distance ≤ 3 in T, hence in $G_{L,d}$, hence the corresponding words are at distance $\leq 3d$ for δ_{pp}.

Using this for infinite languages

Definition

For L a language and $i, \ell \in \mathbb{N}$, define the *i-th ℓ-stratum* of L as

$$S_i = \{w \in L \mid (i-1)\ell \leq |w| < i\ell\}$$

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Proposition

Let $L = \mathrm{L}(A)$ with A having only one interchangeable class of loopable states. Let, letting $\ell = 8|A|^2$ and $d = 16|A|^2$, each S_i is d-connected.

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We conclude by concatenating orderings for $S_1, S_2, \ldots$ obtained with the enumeration technique of the previous slide, with carefully chosen starting and ending points (BLACKBOARD).

Conclusion

Main results (Levenshtein and push-pop)

Let L be regular. Then:

- There exists $t \in \mathbb{N}$ and regular languages $L_1, \ldots, L_t$ such that

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  - → This shows L is orderable for Levenshtein iff it is for push-pop!
- When L is orderable for push-pop then, in a suitable pointer machine model, we have an algorithm that outputs push-pop edit scripts to enumerate L, with constant delay (i.e., independent from the current word length)

Other results and future work

Other results:

- It is *NP*-hard, given a DFA A such that $\mathrm{L}(A)$ is orderable (for Levenshtein or push-pop), to determine the minimal d such that $\mathrm{L}(A)$ is d-orderable.
- A regular language is partitionable into finitely many orderable languages for the push-pop-right distance if and only if it is slender.

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Open and future work:

- Make the delay polynomial in $|A|$? (currently it is exp)
- Implementation and real-life use-cases?

Thanks for your attention!