# Weighted Counting of Matchings in Unbounded-Treewidth Graph Families

Antoine Amarilli, **Mikaël Monet**

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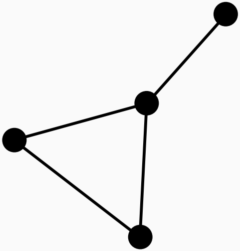
# Who

Joint work with Antoine Amarilli

https://arxiv.org/abs/2205.00851
(to appear in MFCS'22)

# What

- Matching in a graph: set of edges that do not intersect



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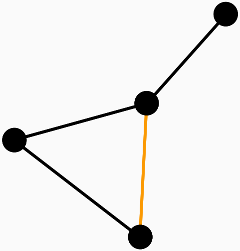

OK

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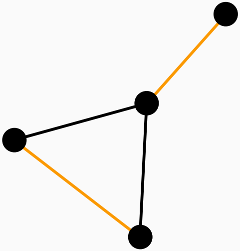

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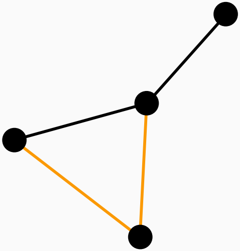

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$\Longrightarrow$ Is there a more general criterion than bounded treewidth that allows matchings to be counted efficiently? **No!***

* subject to defining the problem in a slighlty more general way and assuming certain "treewidth-constructibility" requirement; see next slide for Proper Usage.

# Our result

**Theorem**

Let $\mathcal{G}$ be an arbitrary family of graphs which has unbounded treewidth. Then the problem, given a graph $G = (V, E)$ of $\mathcal{G}$, of computing the number of matchings of $G$, is intractable.

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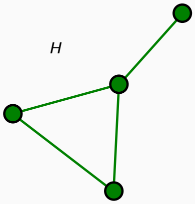
- probability of a matching in $G$: probability of drawing a matching when we select each edge independently with probability $\pi(e)$
- treewidth-constructible: given $k \in \mathbb{N}$ as input, we can construct in polynomial time a graph of $\mathcal{G}$ whose treewidth is $\geq k$

# Proof sketch

We reduce from counting matchings on planar graphs of maximum degree 3, which is #P-hard. Let $H$ be such a graph.



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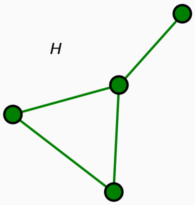
- **Step 1.** Using treewidth constructibility of $\mathcal{G}$ and the polynomial grid-minor extraction algorithm of [Chekuri and Chuzhoy, 2016], construct a graph $G \in \mathcal{G}$ such that $H$ is a topological minor of $G$



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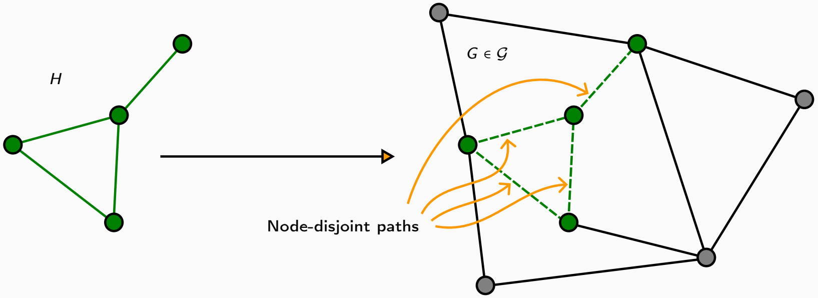

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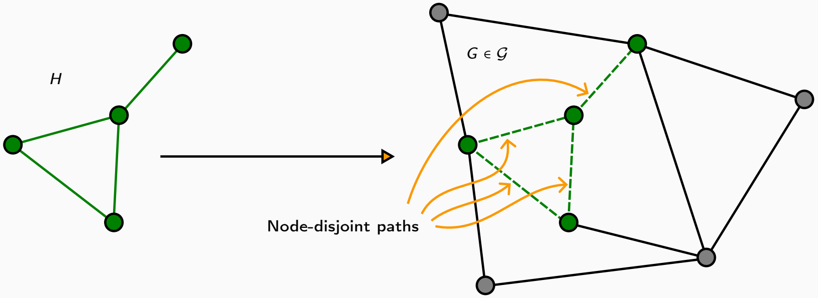

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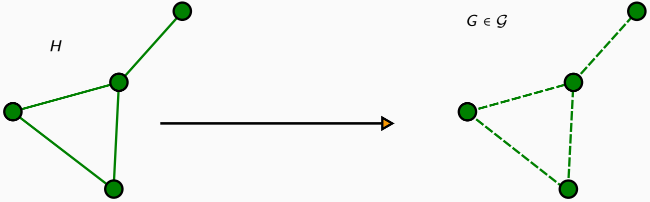

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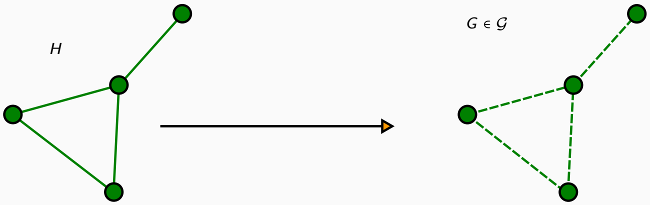


- **Step 3.** Somehow, use polynomial interpolation

# Technical challenge

“Emulate” long paths with probability 1/2 with short paths:

Find $p, q, r, s \in [0; 1]$ such that the probability of a matching in


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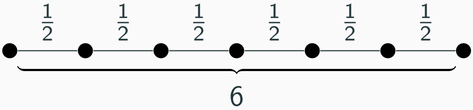

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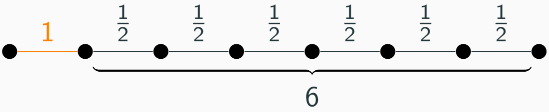

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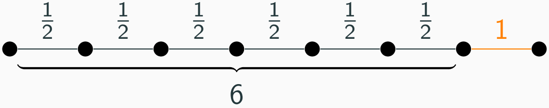

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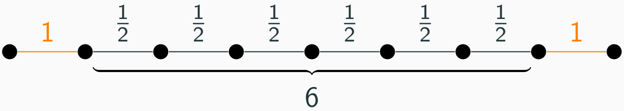

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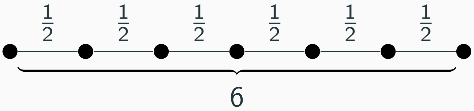

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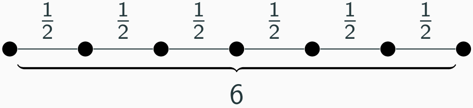


$$(p,q,r,s) = \left(\tfrac{1}{4992}\sqrt{1002921} + \tfrac{977}{1664},\ \tfrac{3}{7600}\sqrt{1002921} + \tfrac{3367}{7600},\ -\tfrac{3}{7600}\sqrt{1002921} + \tfrac{3367}{7600},\ -\tfrac{1}{4992}\sqrt{1002921} + \tfrac{977}{1664}\right)$$

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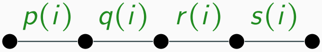


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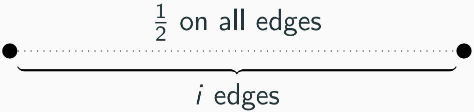


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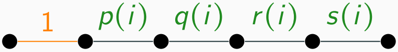


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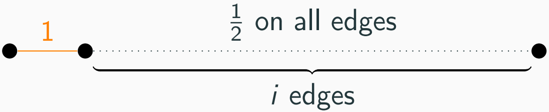


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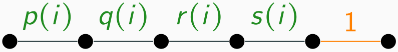


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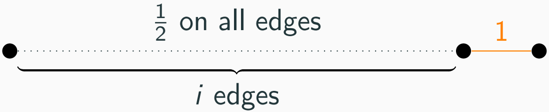


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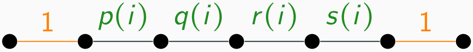


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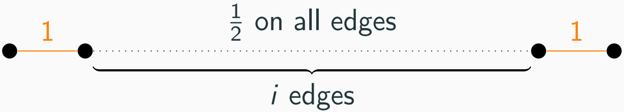


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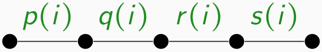


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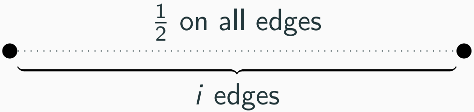


$(p(i), q(i), r(i), s(i)) = ?$ This is possible when $i$ is even and $\geq 4$

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# Conclusion

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- also holds for edge covers (and most likely also for independent sets and vertex covers, when probabilities are on the nodes)

→ **open:** allow only probabilities in $\{0, 1/2\}$

Thanks for your attention!

Chandra Chekuri and Julia Chuzhoy.
**Polynomial bounds for the grid-minor theorem.**
Journal of the ACM, 63(5):1–65, 2016.