Connecting Width and Structure in Knowledge Compilation

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What is Knowledge Compilation?

- You have a task
  - $\rightarrow$ Boolean SAT (is there a satisfying assignment?)

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  - → Boolean SAT (is there a satisfying assignment?)
  - → #SAT (model counting) (how many satisfying assignments?)
  - → probabilistic evaluation
  - → enumeration
- Idea: compile the input into a format that is *designed* to solve efficiently your task

Why would I do that?

- Without knowledge compilation

$$\text{Input class } \mathcal{C}_1 \xrightarrow{\text{Algo. } \mathbf{1}} \text{Result}$$

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- With knowledge compilation:

$$\text{Input class } \mathcal{C}_1$$
$$\text{Input class } \mathcal{C}_2 \qquad \text{Compilation target for your task} \xrightarrow{\text{Generic algo.}} \text{Result}$$
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Input class $\mathcal{C}_1$ $\xrightarrow{\text{Algo. 1}}$ Result

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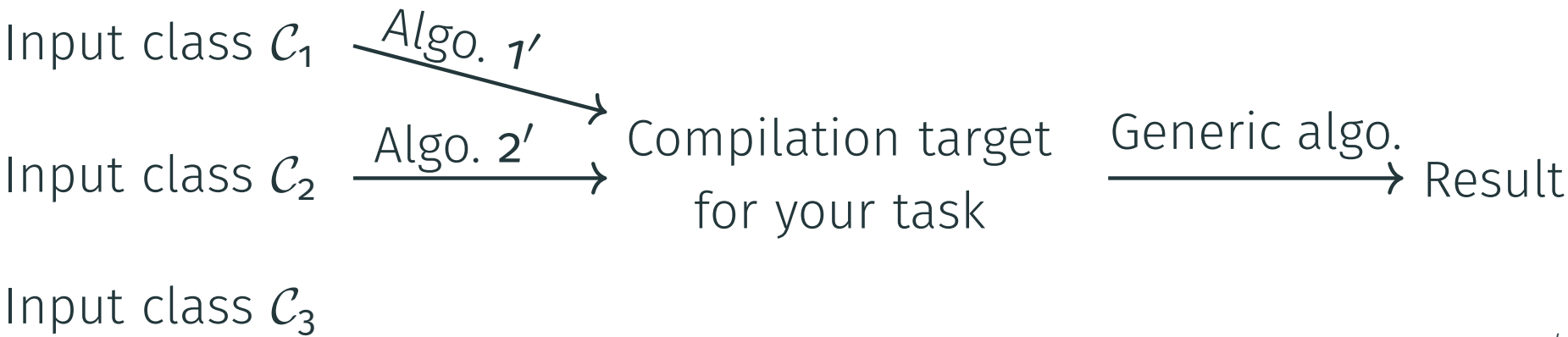

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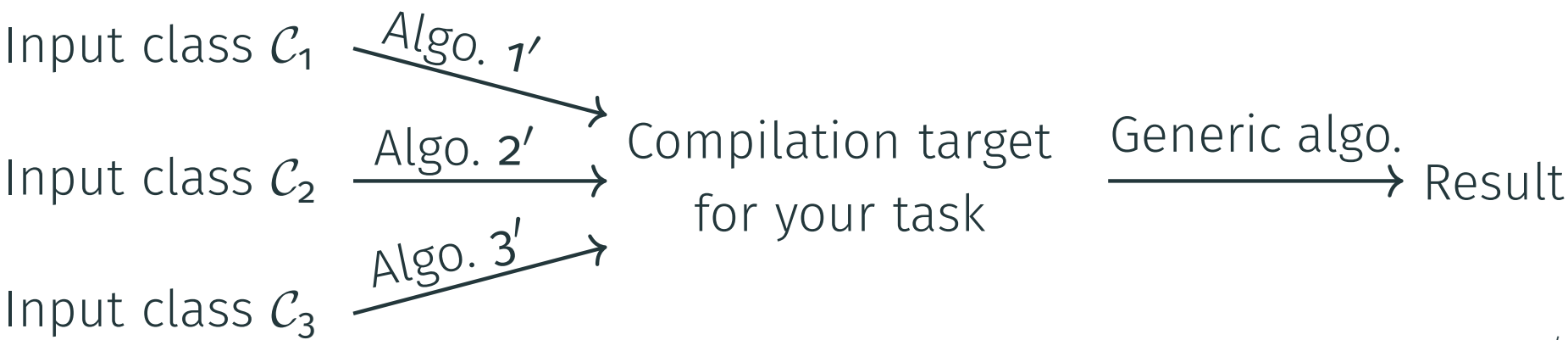

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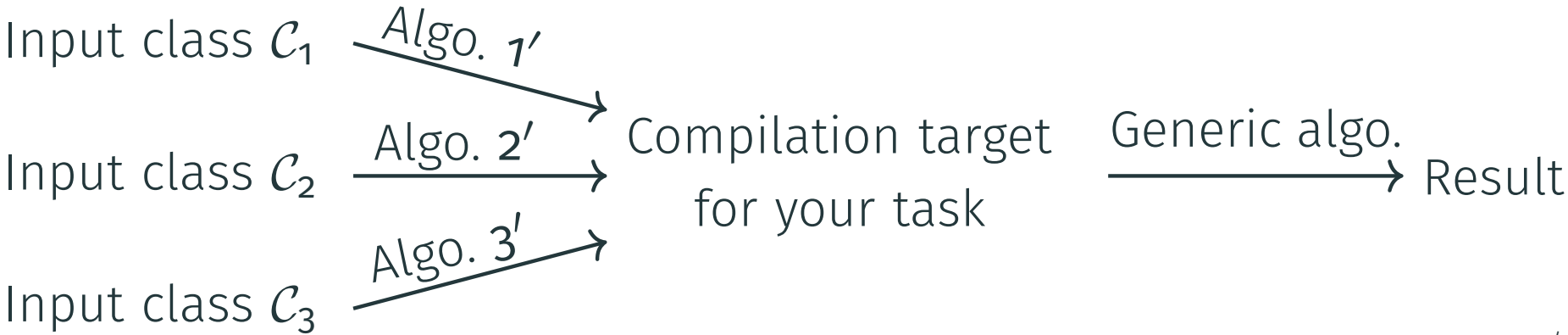
- With knowledge compilation: modularity!



Studying the compilation targets

- Tradeoffs between:
  - $\rightarrow$ Complexity of compilation (*conciseness* of the compilation target)
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$$\text{Input} \xrightarrow{\text{Compilation}} \mathcal{C}_{\text{target}} \xrightarrow{\text{Complexity}} \text{Result}$$

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Truth table

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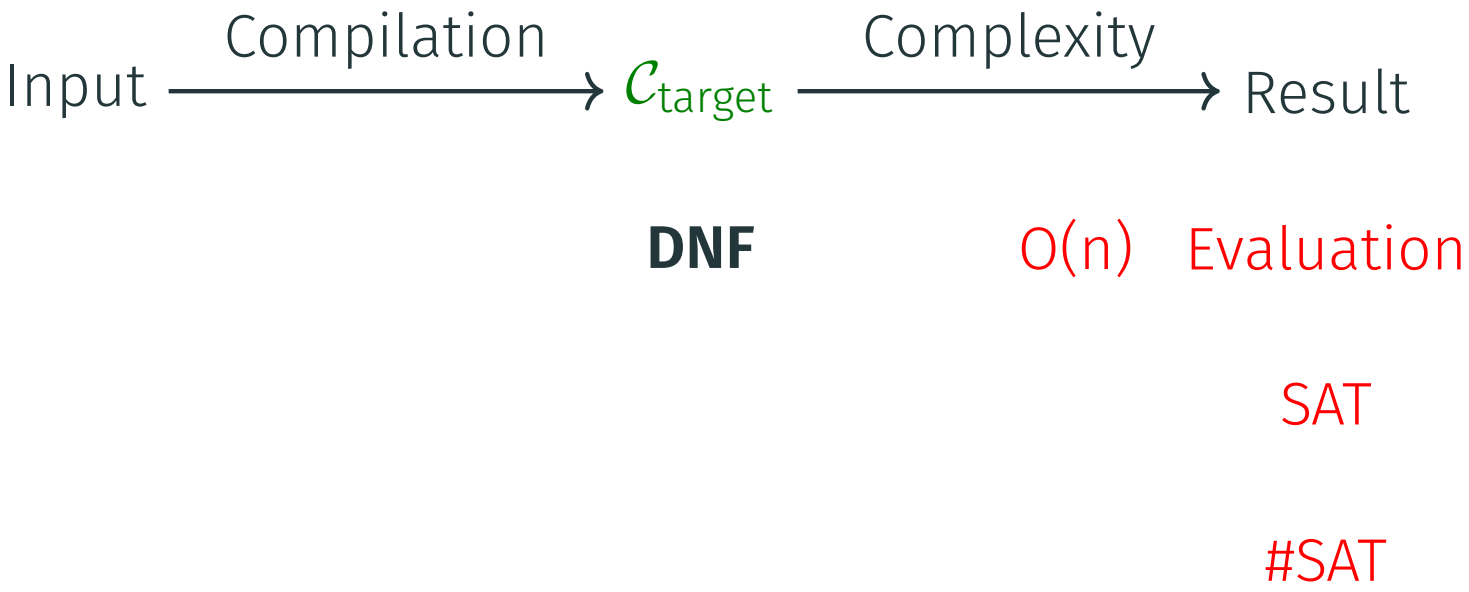

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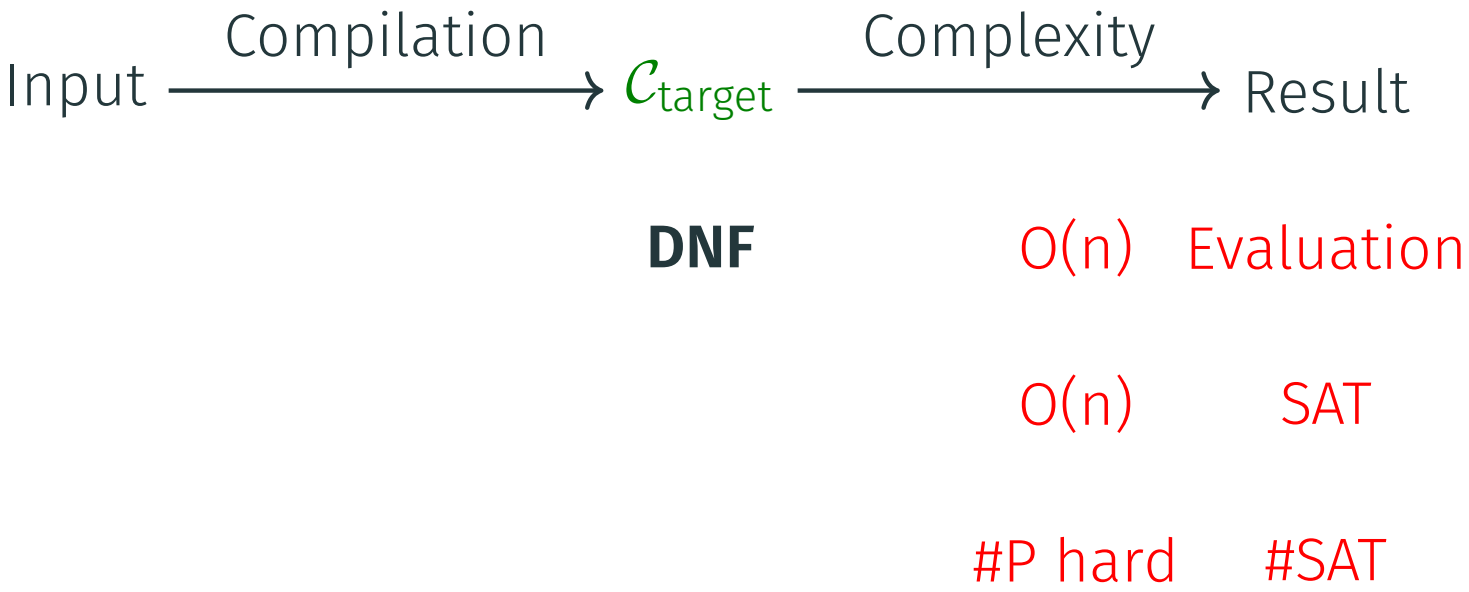

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- We are interested by **#SAT** and **probability evaluation**

Target classes in knowledge compilation

For #SAT and probabilistic evaluation, two main restrictions on compilation targets:

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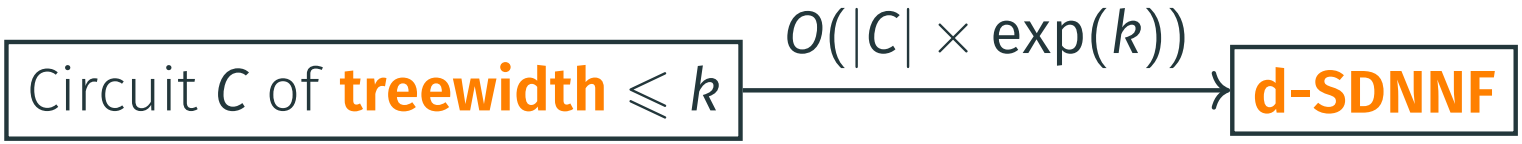
Question: what are the links beetween the two?

Plan

- 

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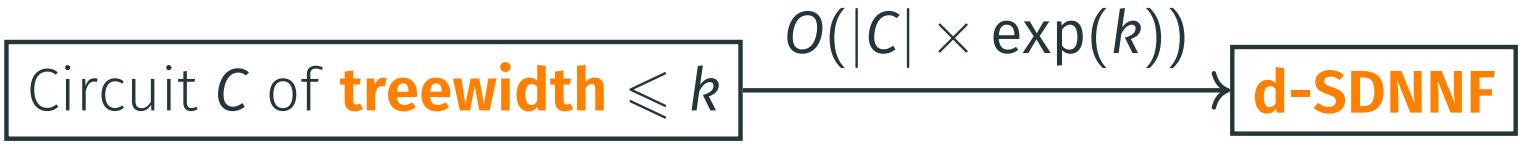
- 

- $+ \simeq$ matching lower bound

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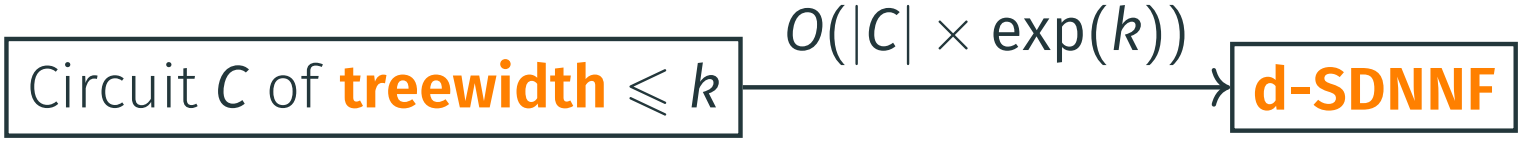
- 

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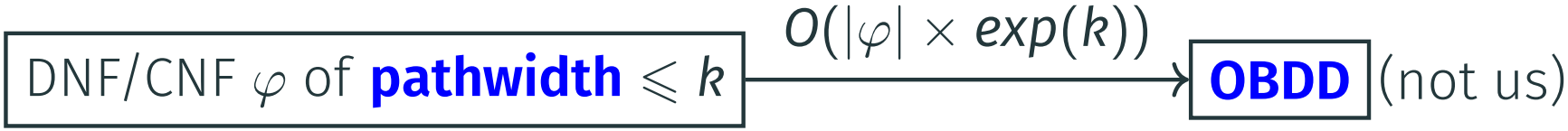
- 

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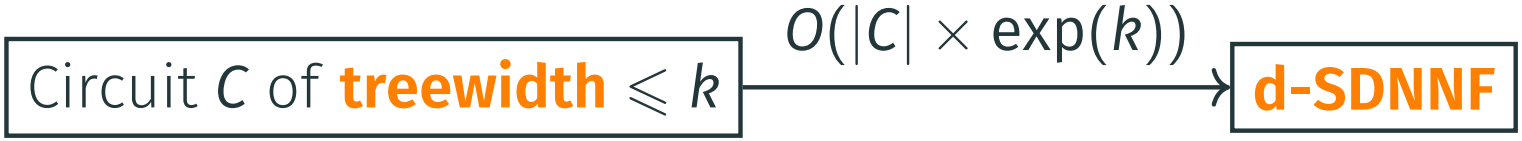
- 

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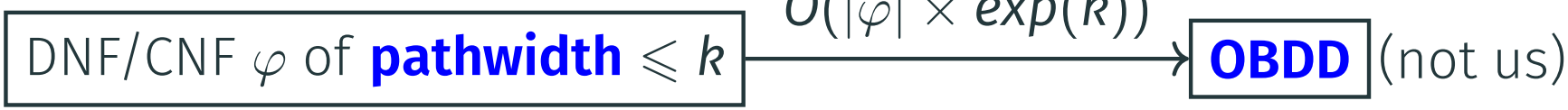
- 

- + matching lower bound

Plan

- Circuit C of **treewidth** $\leqslant k$ $\xrightarrow{O(|C| \times \exp(k))}$ **d-SDNNF**
- + $\simeq$ matching lower bound

Then

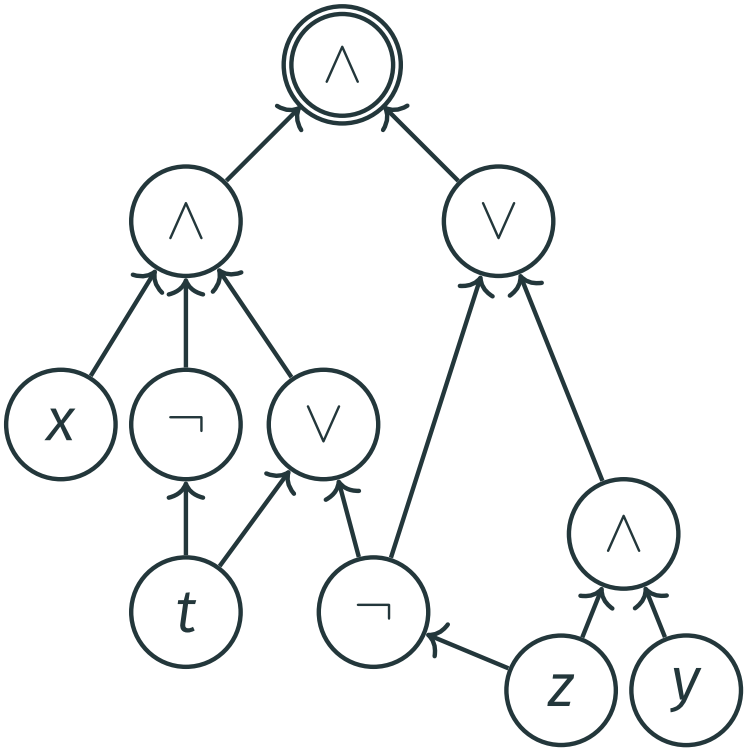
- DNF/CNF φ of **pathwidth** $\leqslant k$ $\xrightarrow{O(|\varphi| \times exp(k))}$ **OBDD** (not us)
- + matching lower bound

Then

- Application to provenance and probabilistic databases

Treewidth and **d-SDNNFs**

Bounded treewidth Boolean circuits



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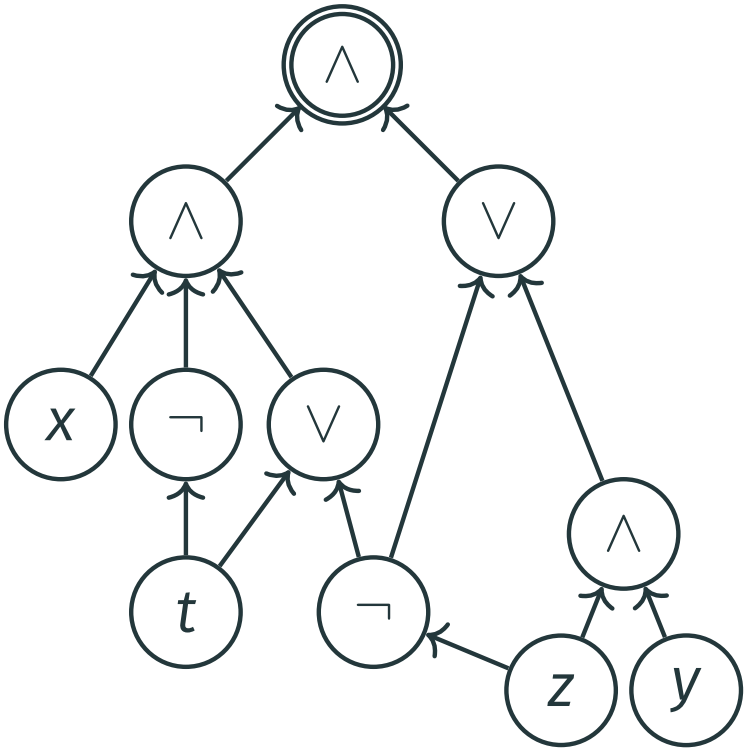

Treewidth of C = that of the underlying graph

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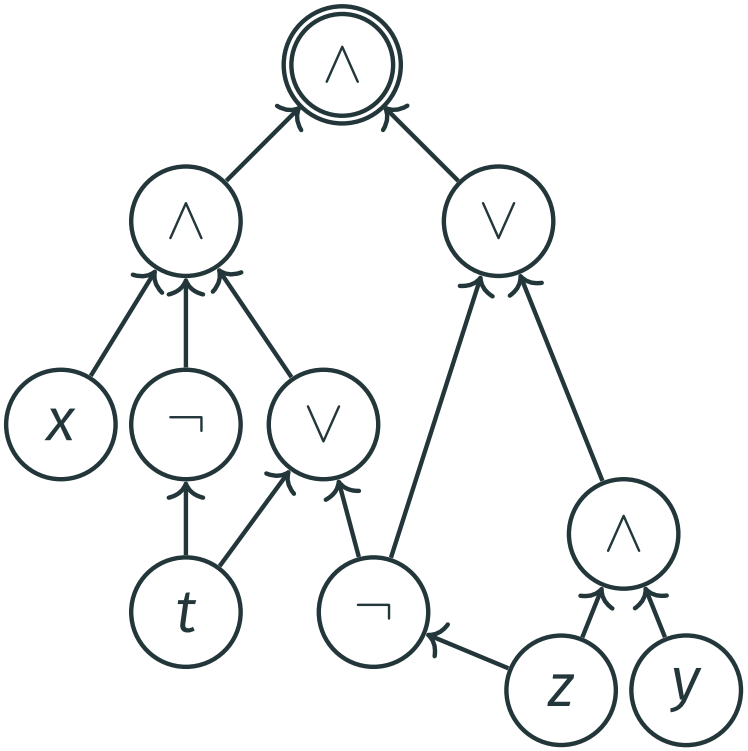


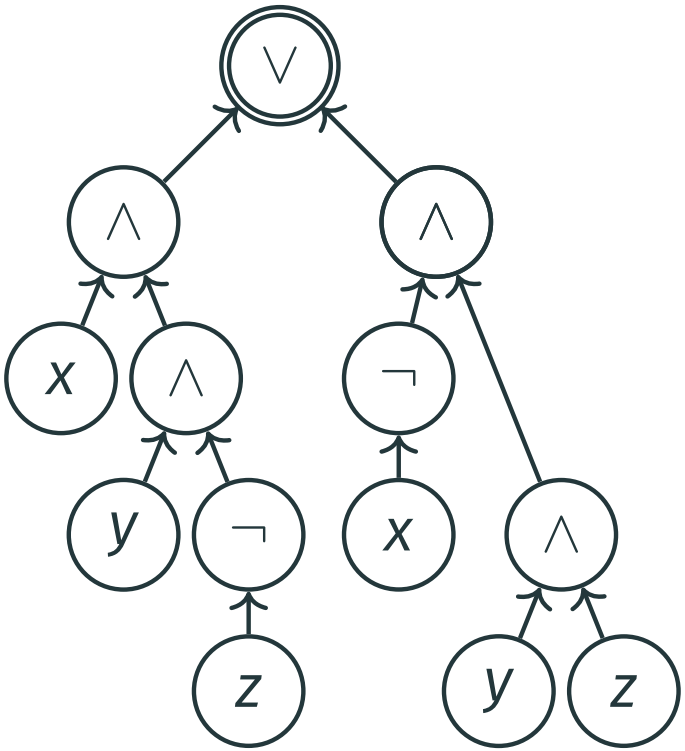
Treewidth of C = that of the underlying graph

We can do message passing:

Theorem (Lauritzen & Spielgelhalter, 1988)

Fix $k \in \mathbb{N}$. Given a Boolean circuit C of treewidth $\leqslant k$, we can compute its probability in time $O(f(k) \times |C|)$, where f is singly exponential

d-SDNNF



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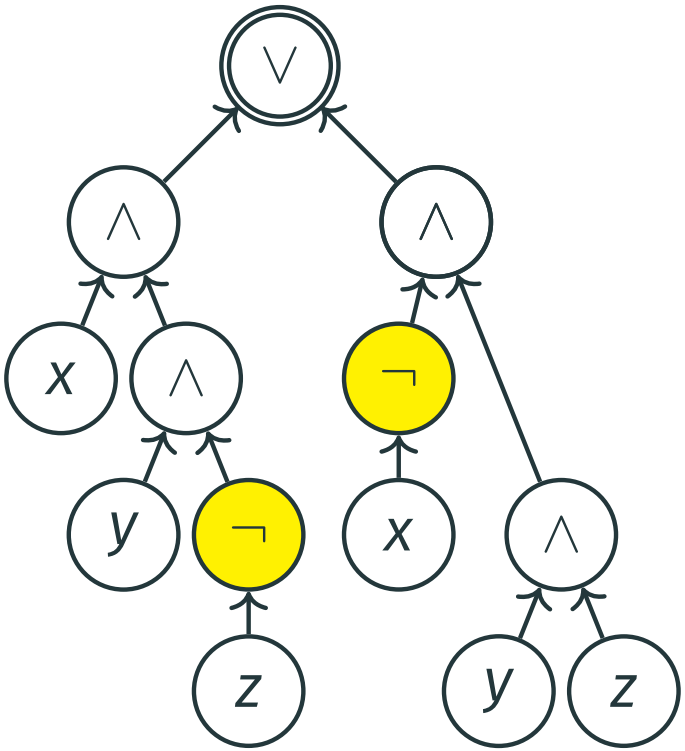
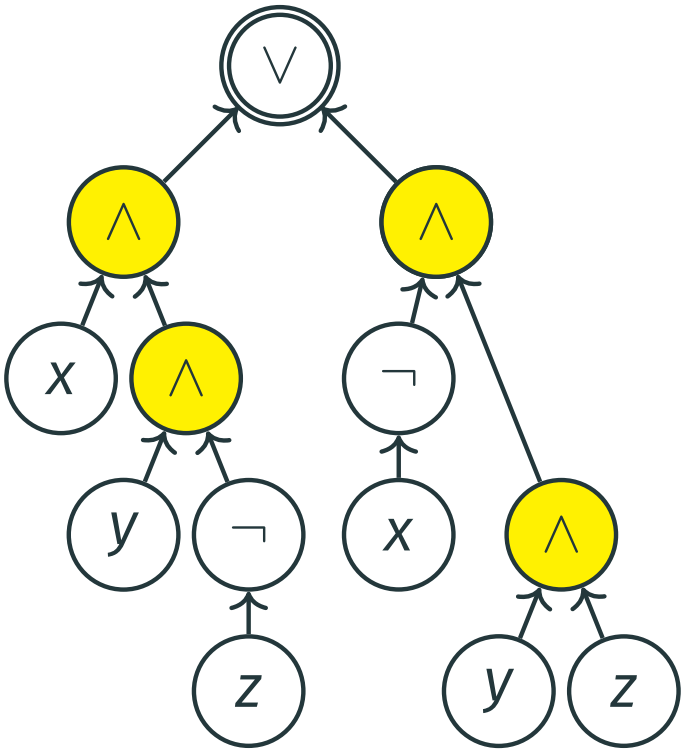


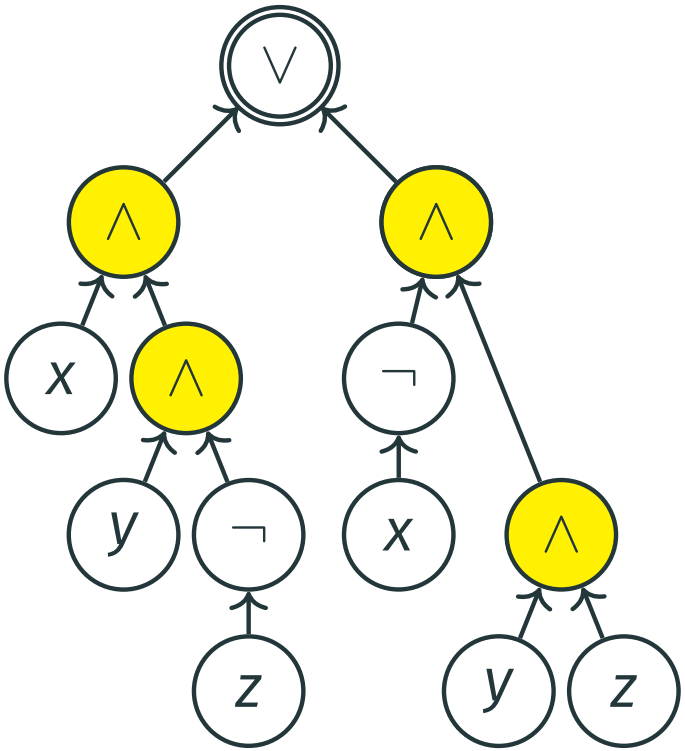
- Negation Normal Form: negations only applied to the leaves

d-SDNNF



- Negation Normal Form: negations only applied to the leaves
- Decomposable: inputs of $\wedge$-gates are **independent** (no variable has a path to two different inputs of the same $\wedge$-gate)

d-SDNNF



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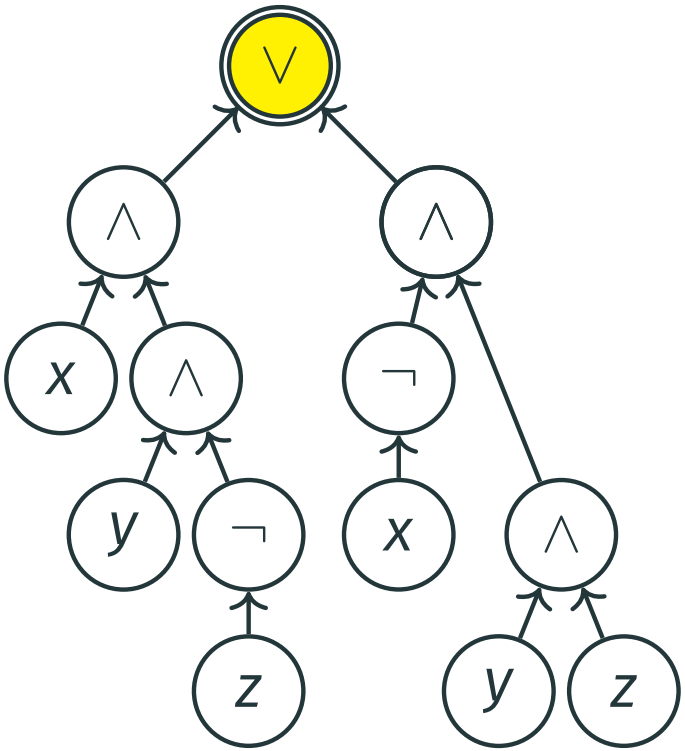

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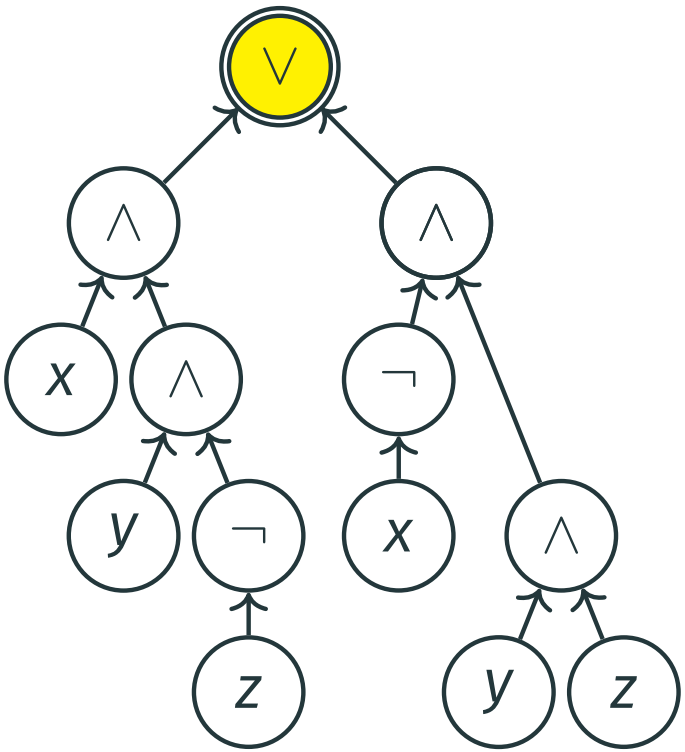

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  - $\rightarrow$ **#SAT** and **probability evaluation**

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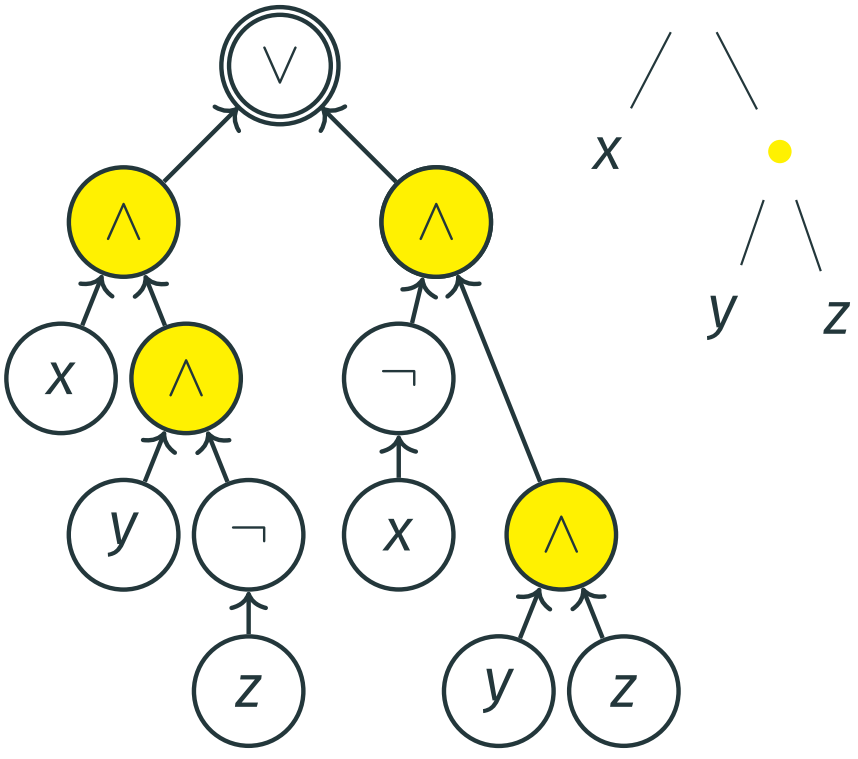

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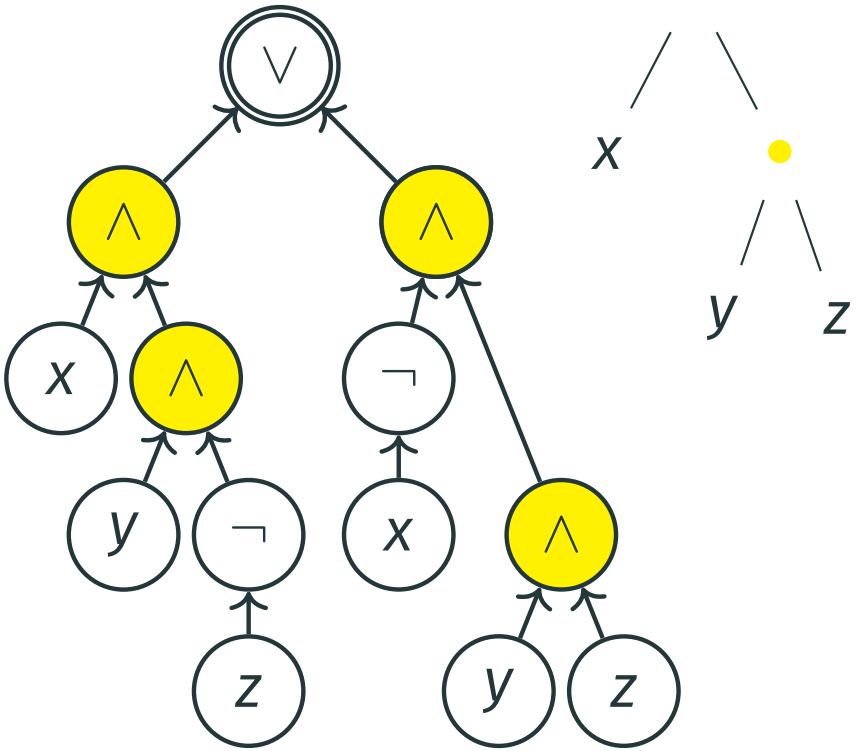

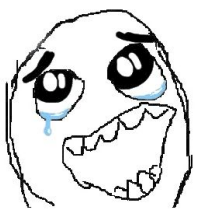

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  - $\rightarrow$ **#SAT** and **probability evaluation**
- Structured: there is a **vtree** that structures the $\wedge$-gates
  - $\rightarrow$ **Enumeration**

Treewidth and d-SDNNFs: Upper bound

Theorem (Bova & Szeider, 2017)

Let C be a Boolean circuit on m variables of ***treewidth*** $\leqslant k$.
There exists *a* ***d-SDNNF*** *equivalent to C of size $O(m \times g(k))$,*
where g is ***doubly*** *exponential*

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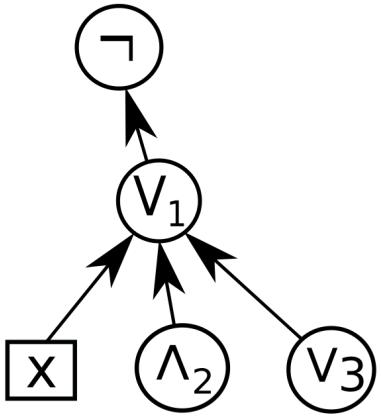
Applications: recapturing message passing, and enumeration of satisfying valuations

Construction sketch



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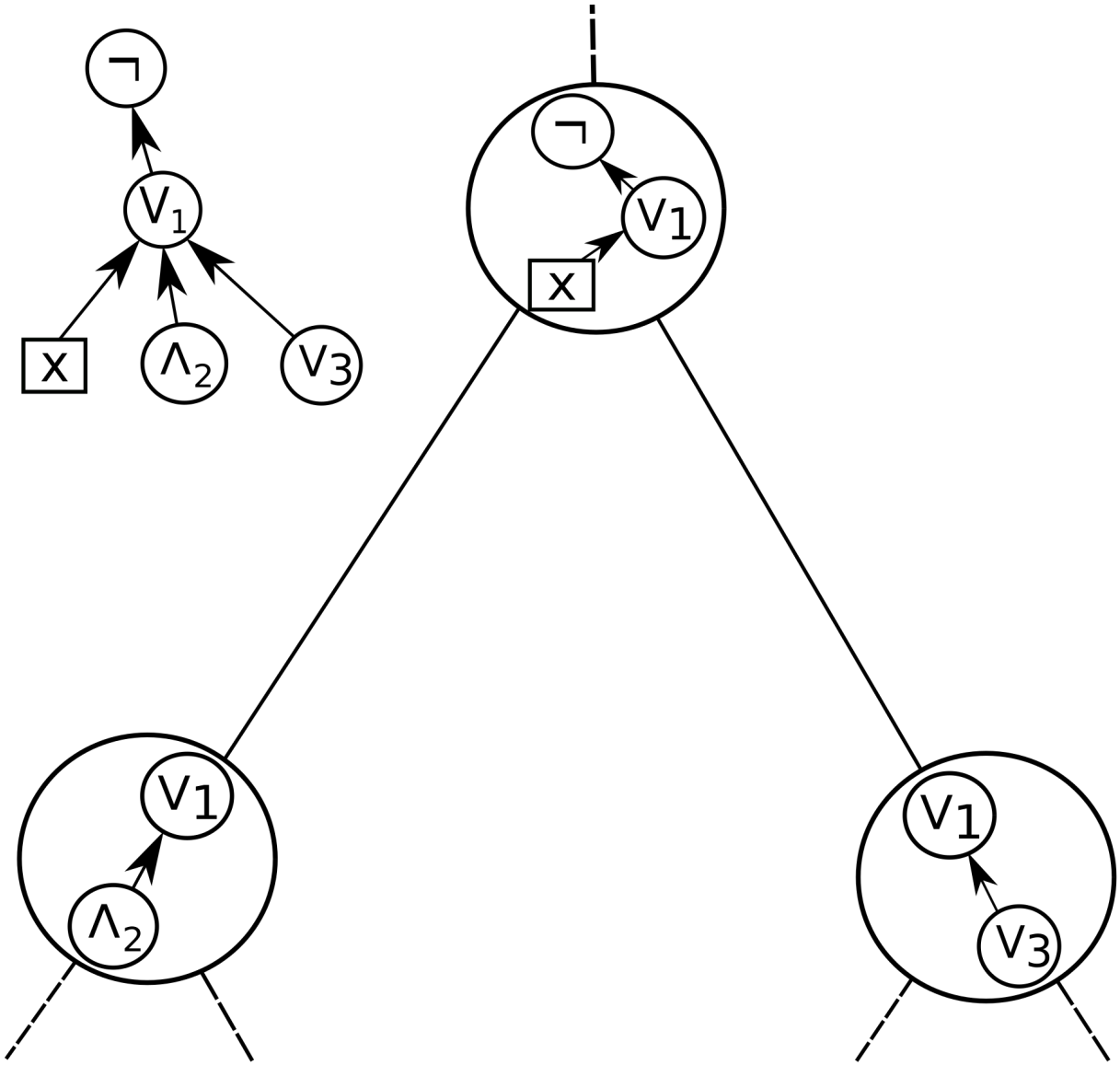

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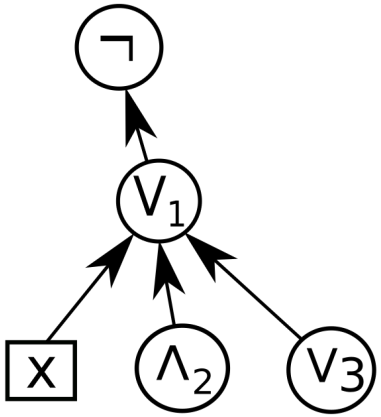

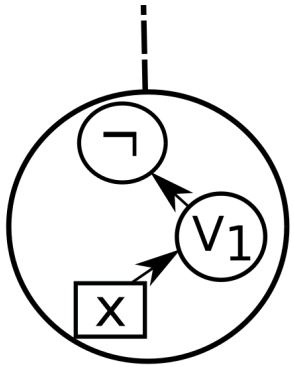

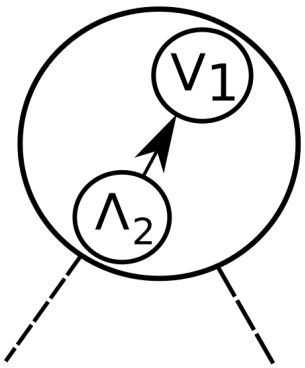

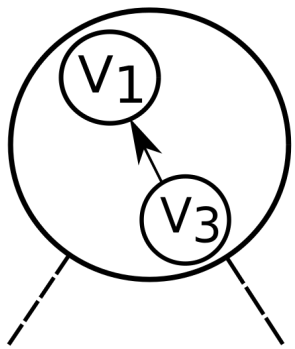

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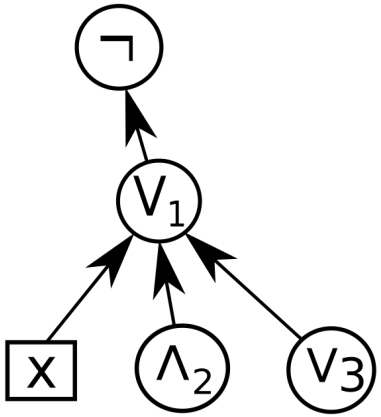

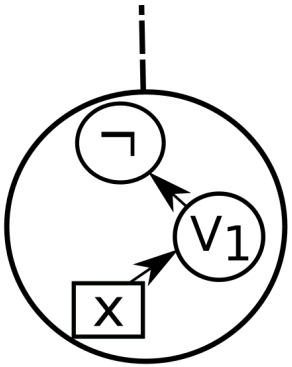

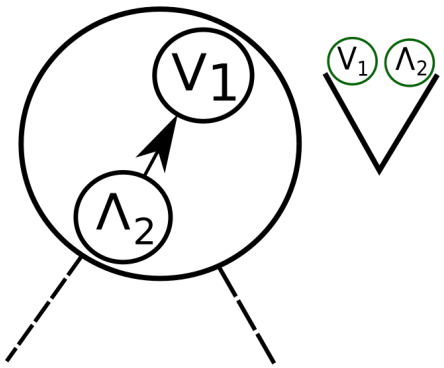

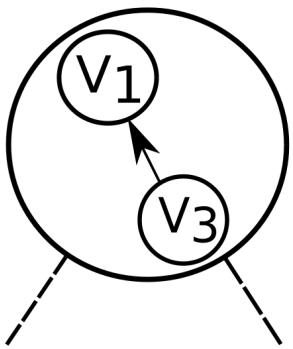

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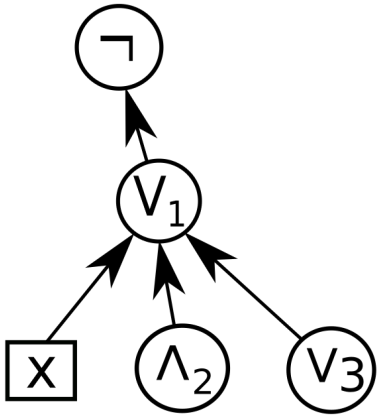

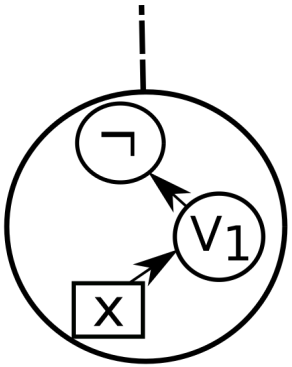

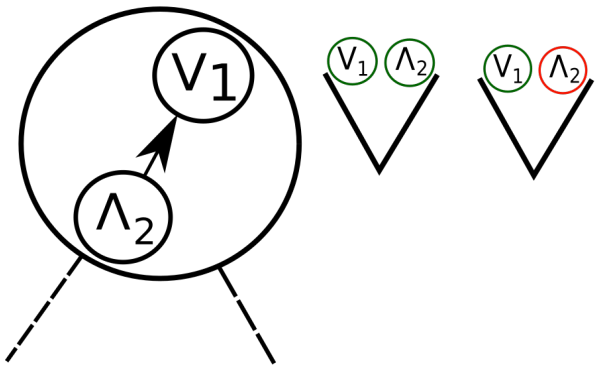

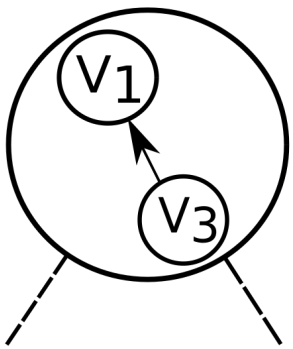

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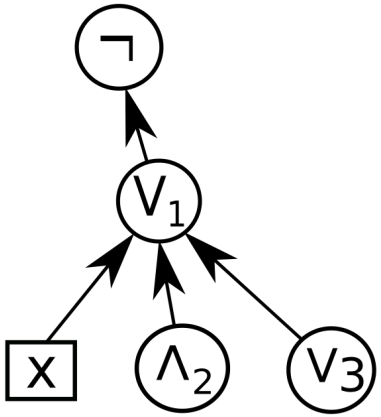

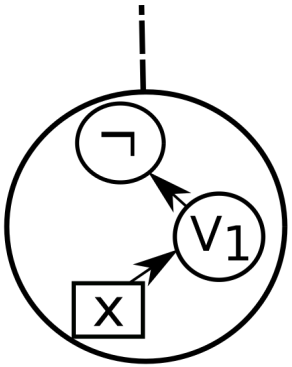

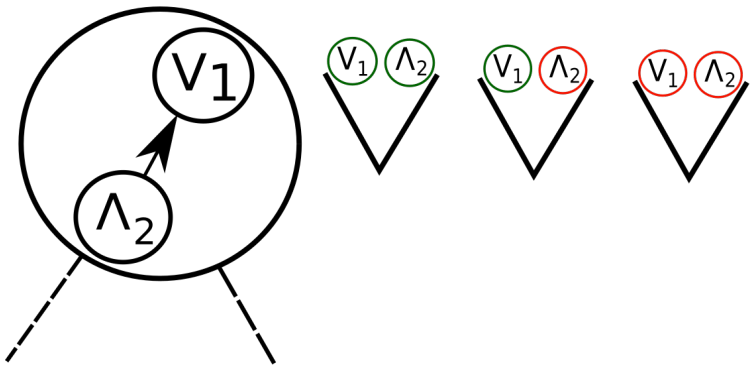

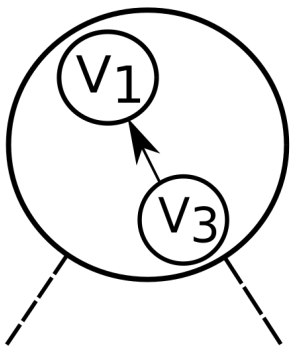

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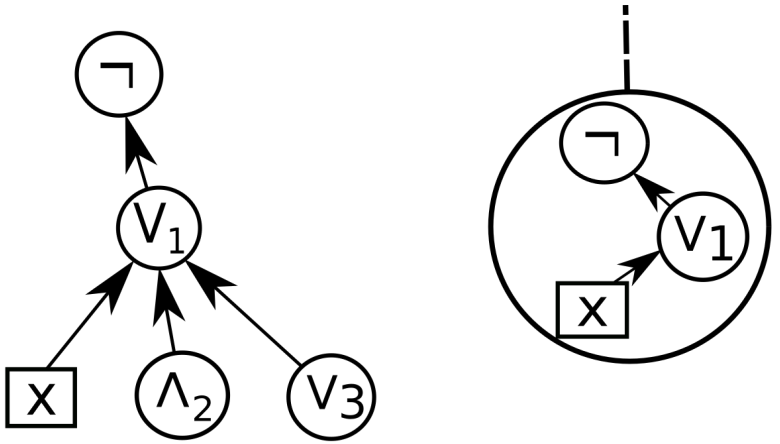

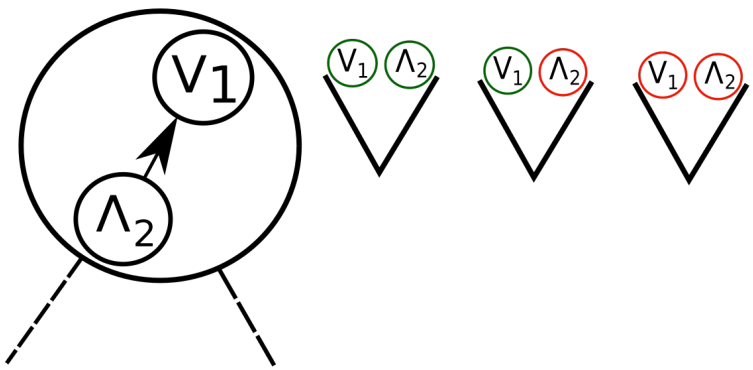

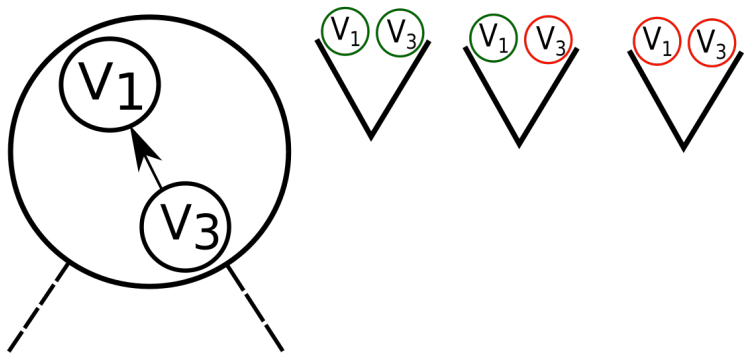

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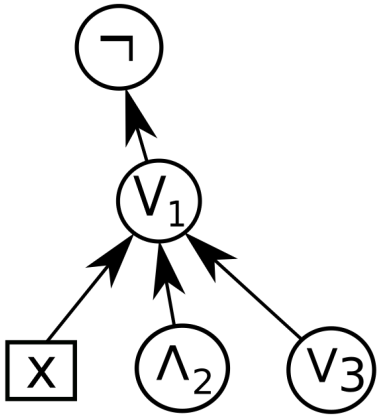

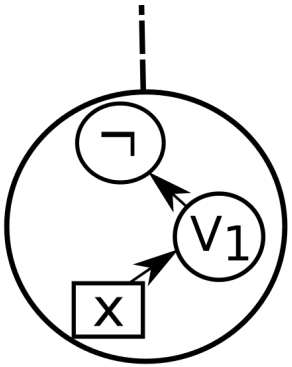

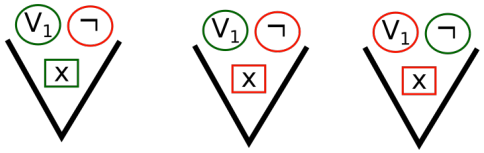

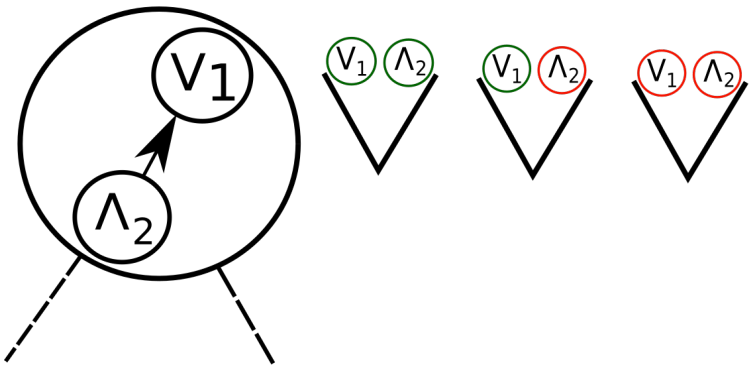

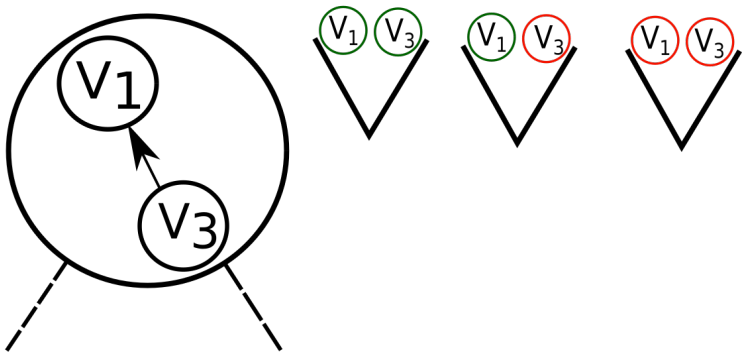

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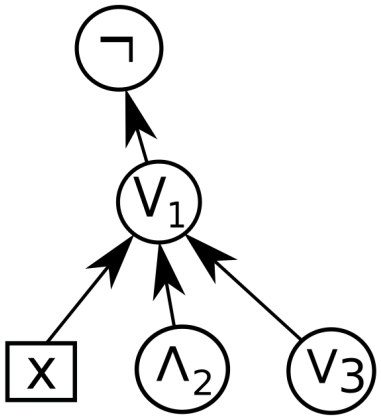

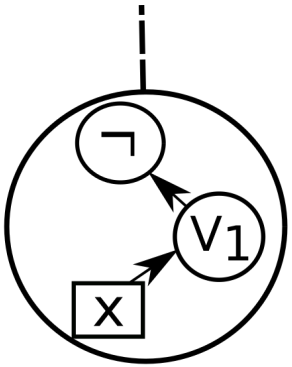

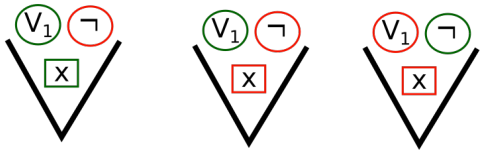

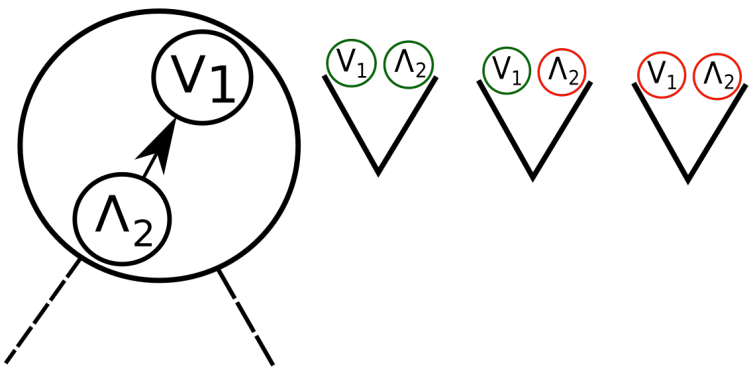

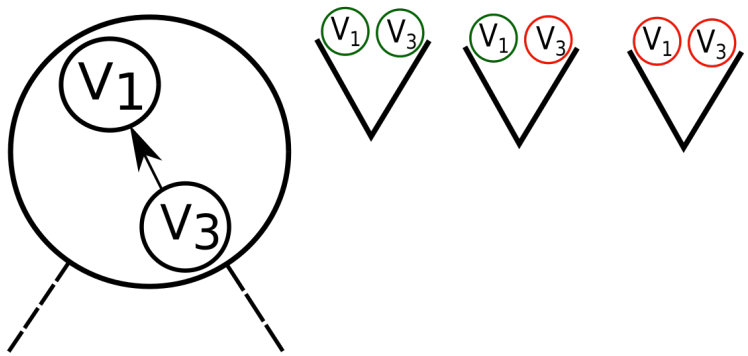

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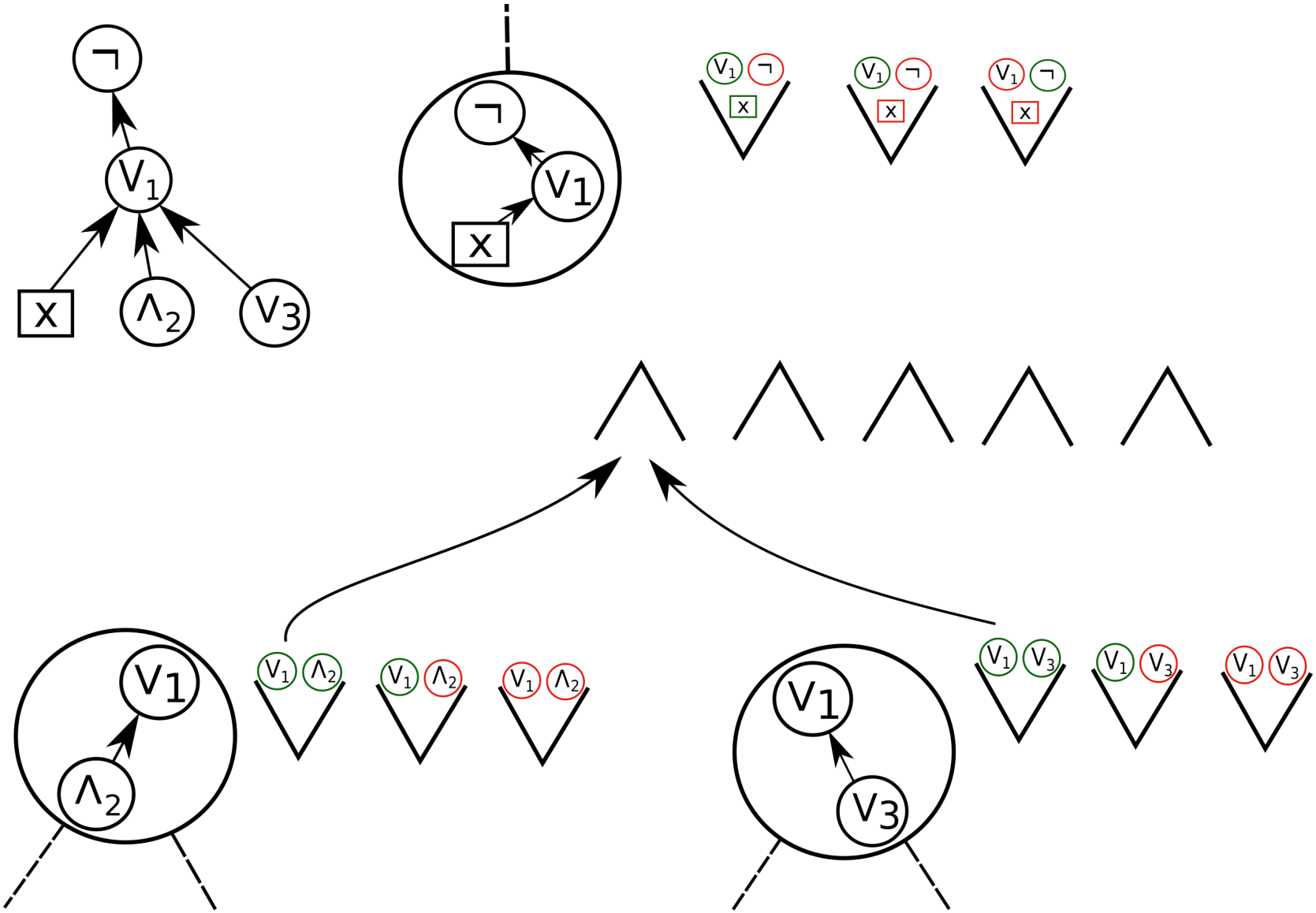

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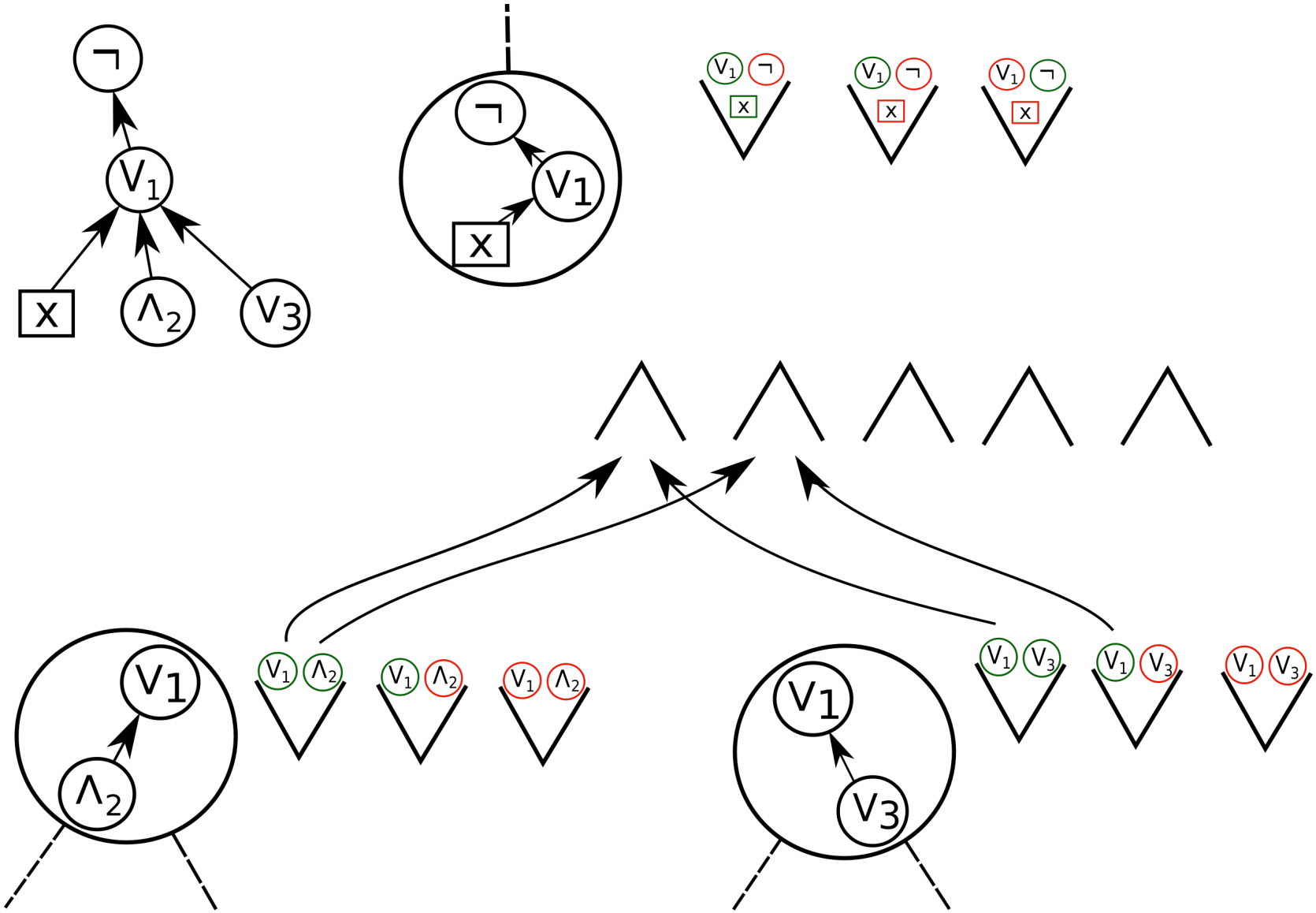

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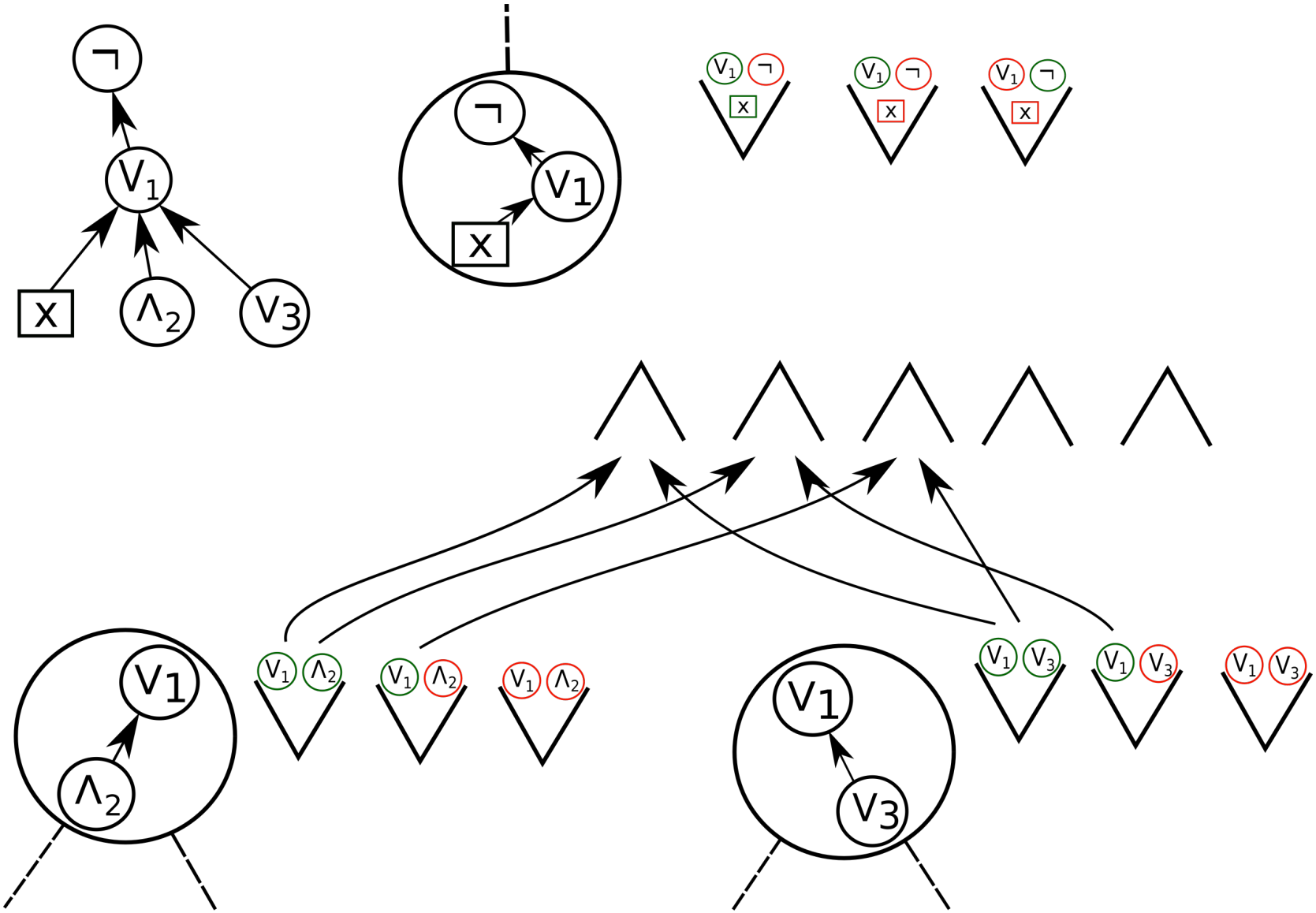

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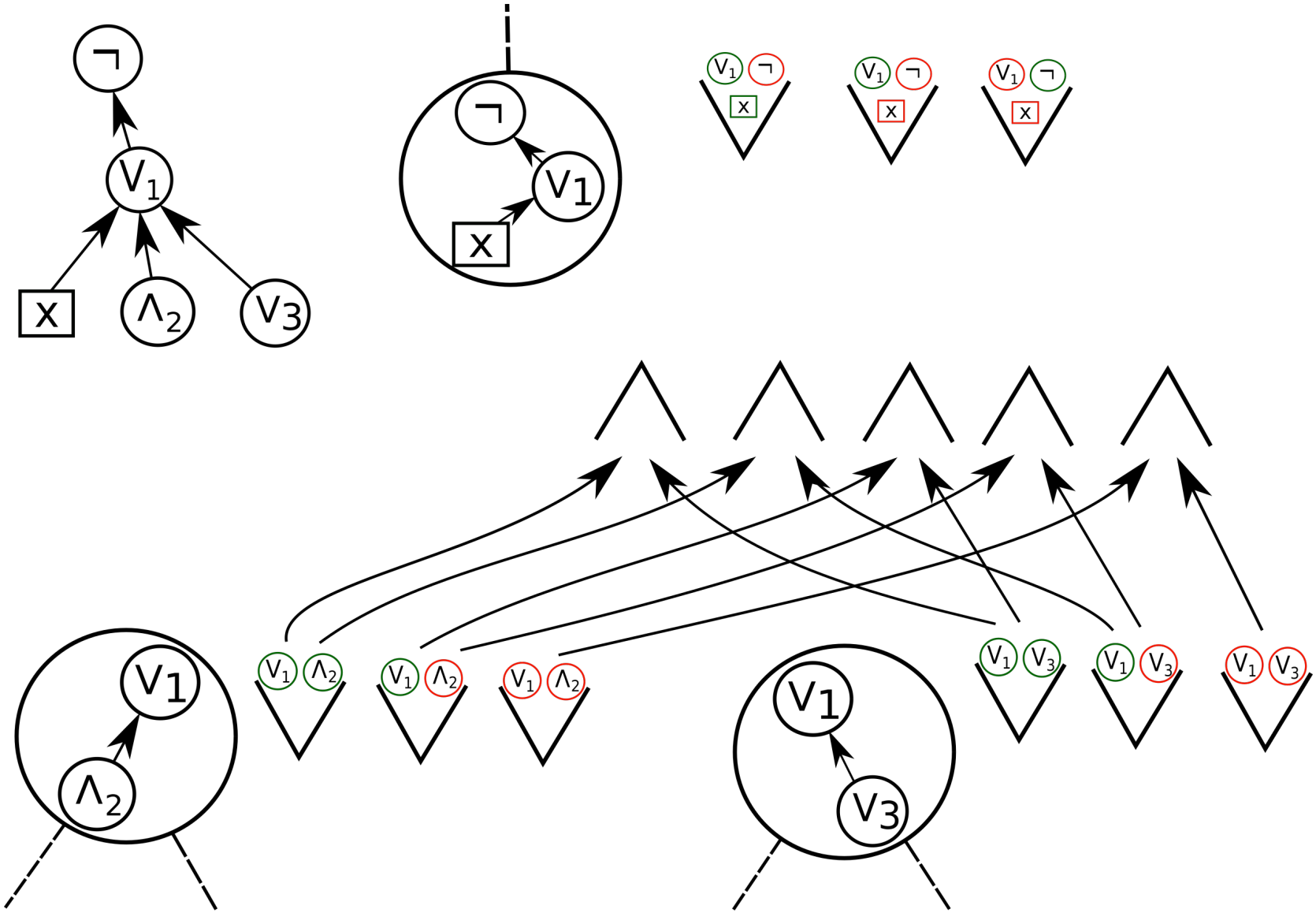

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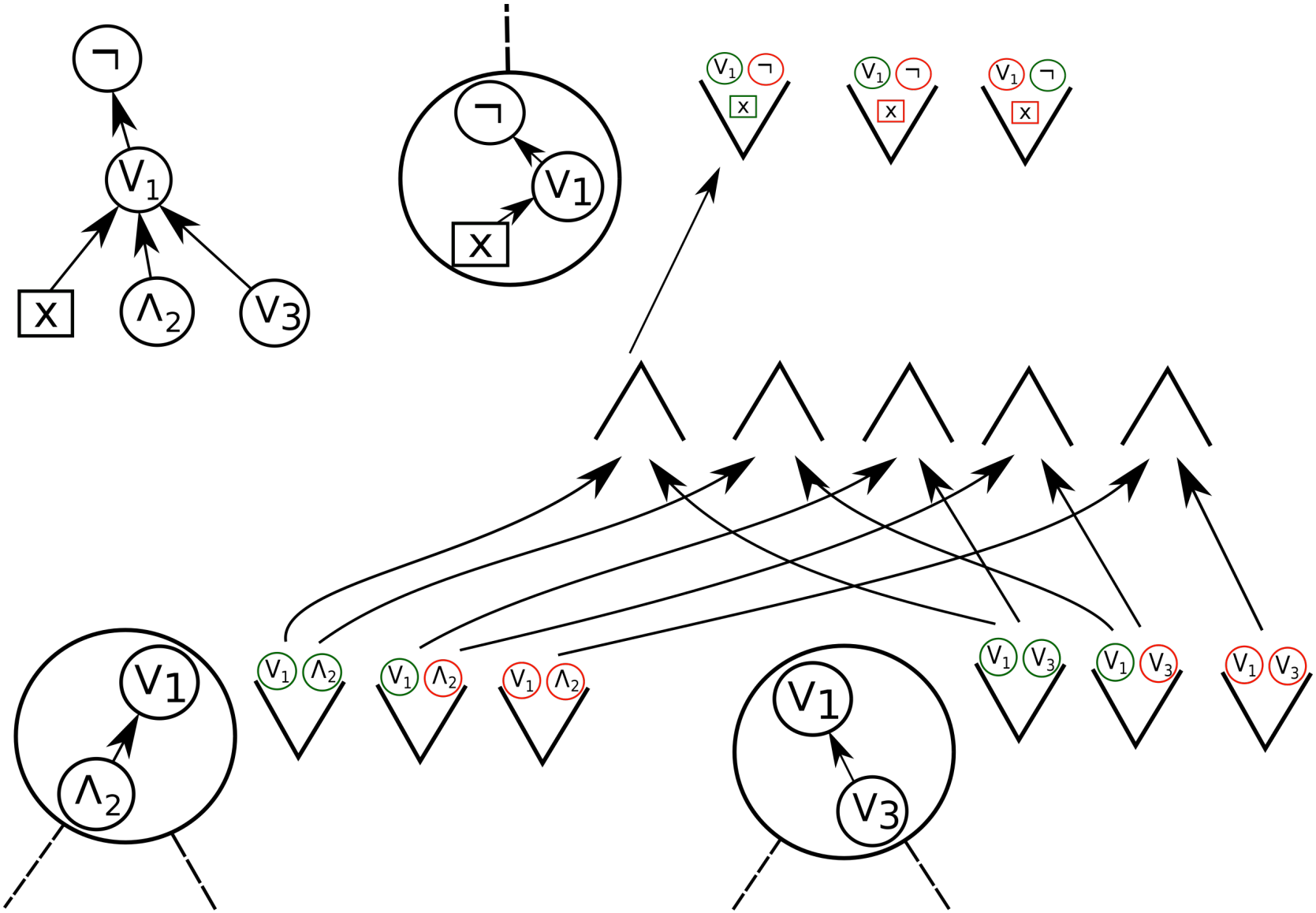

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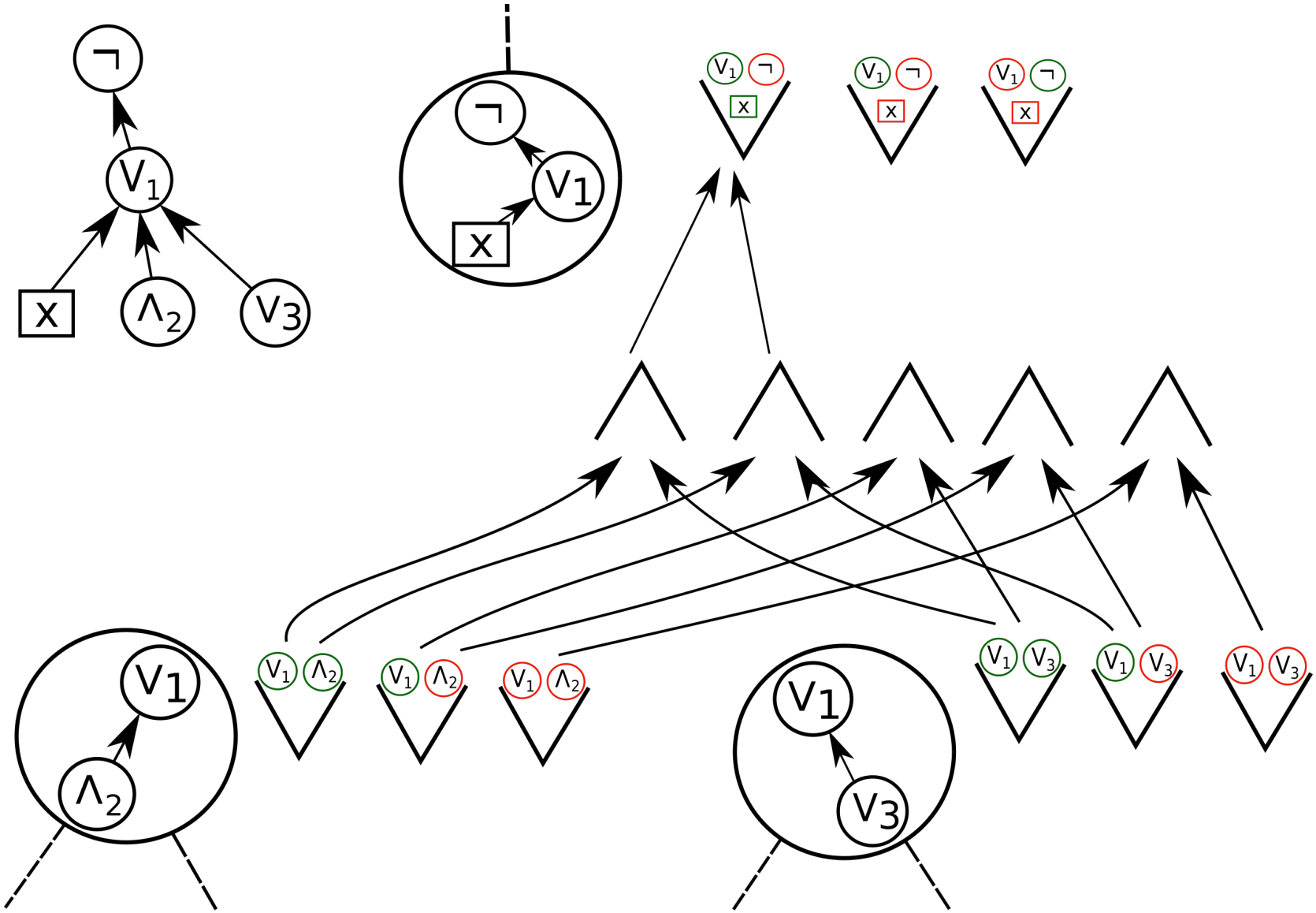

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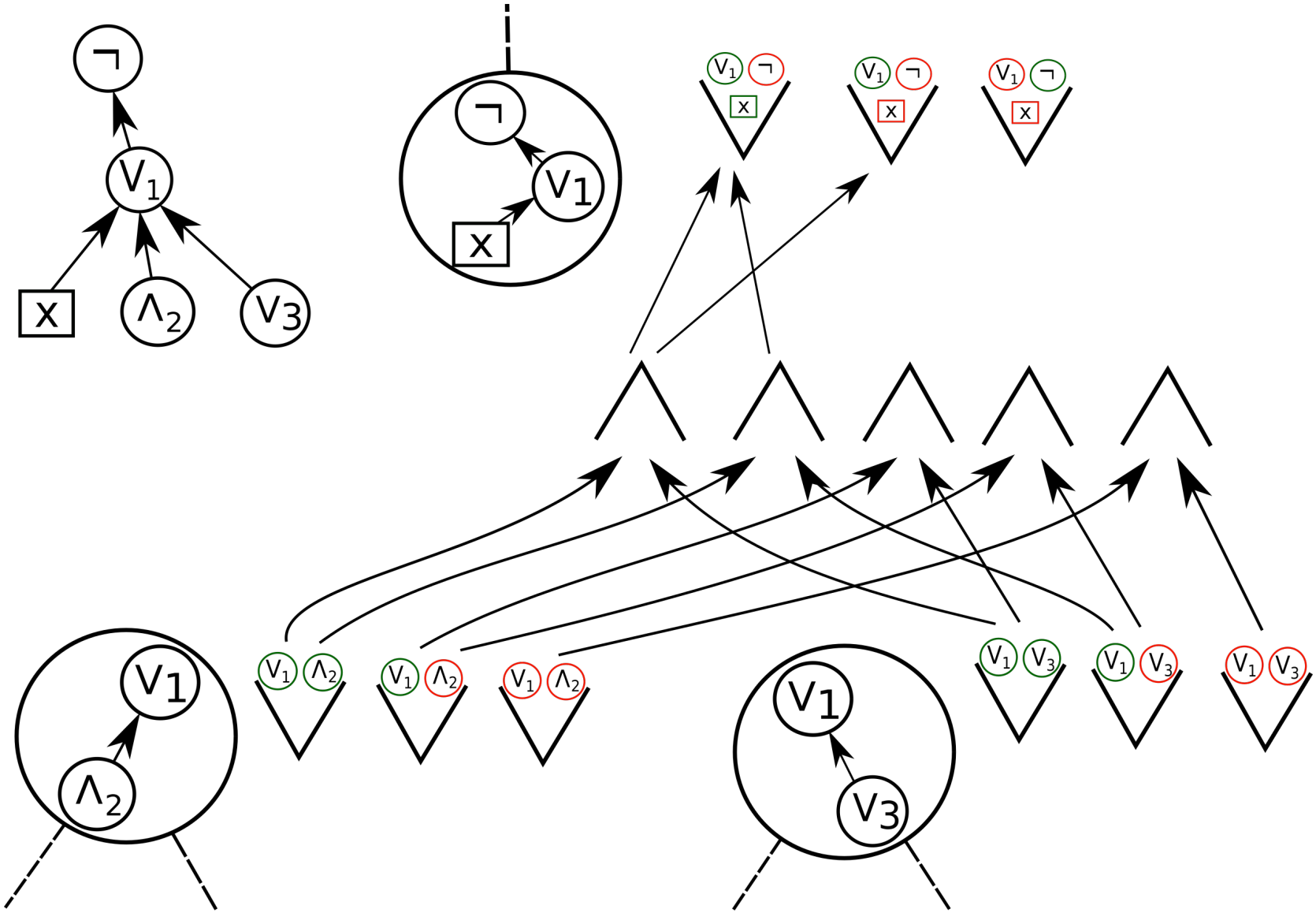

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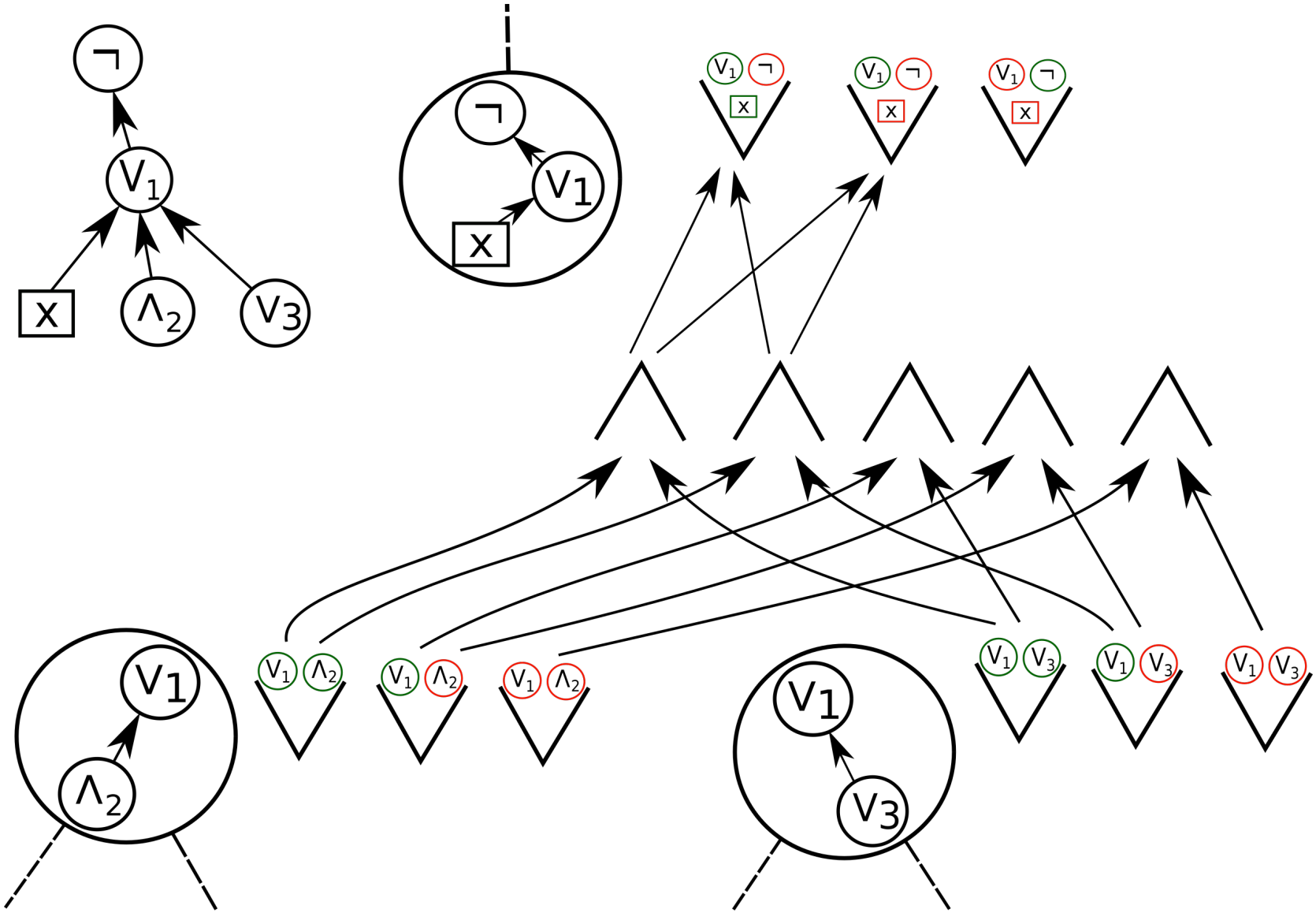

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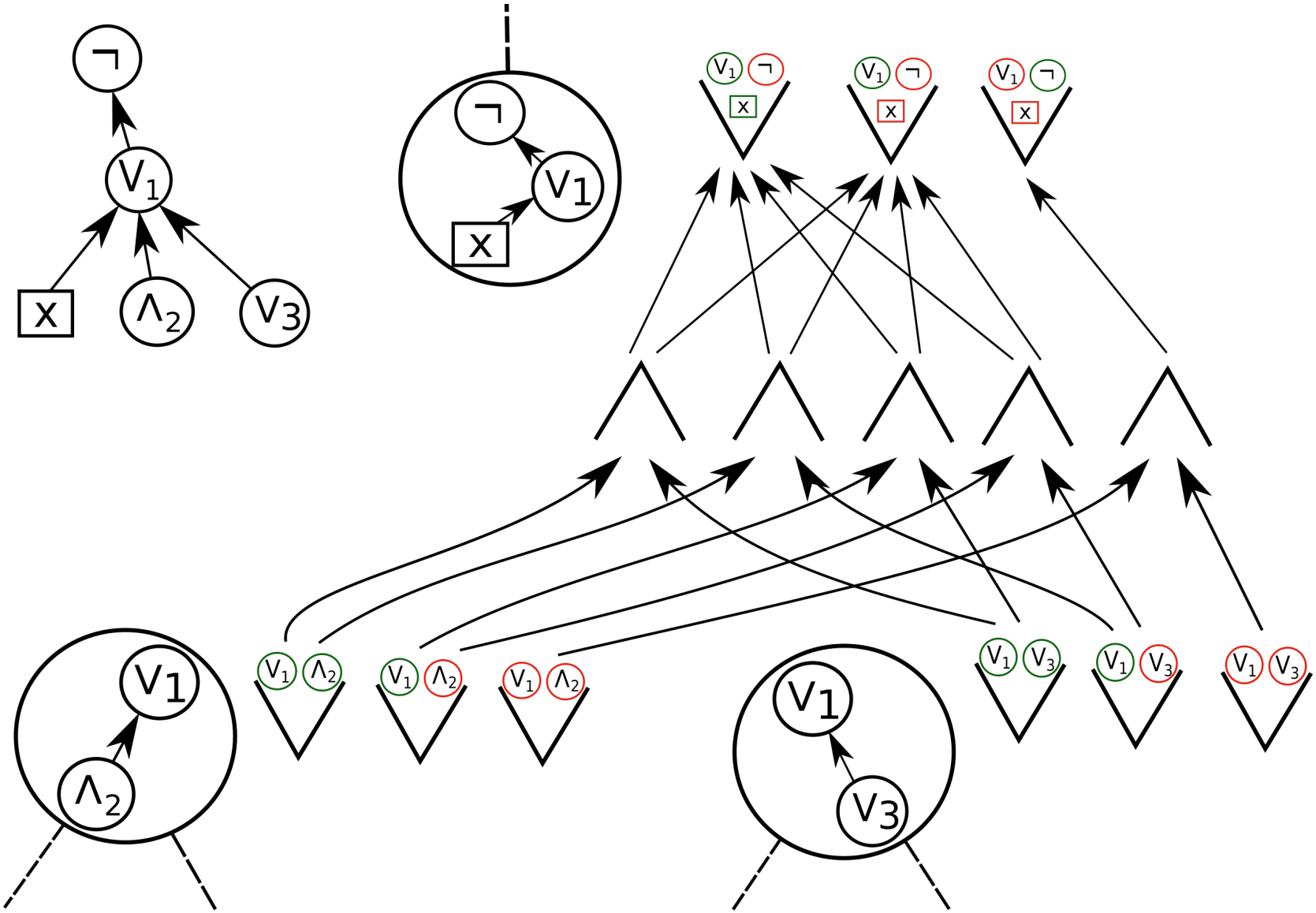

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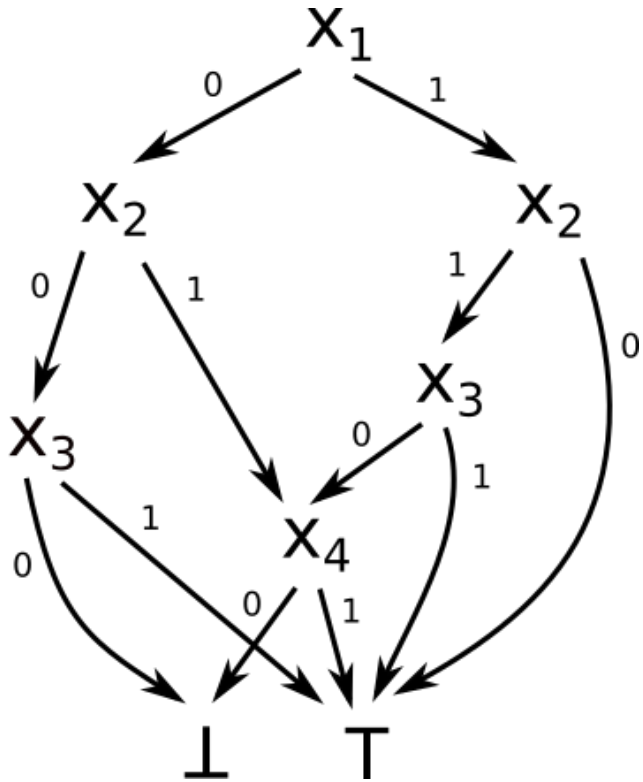
Pathwidth and **OBDDs**

Ordered Binary Decision Diagrams (OBDDs)

- DAG with sink nodes $\{\top, \bot\}$ and internal nodes labeled by variables



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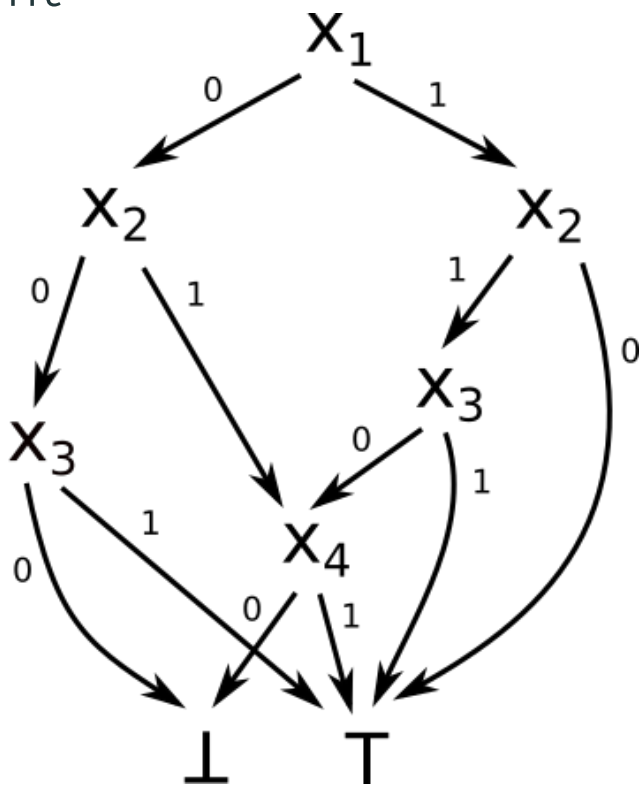

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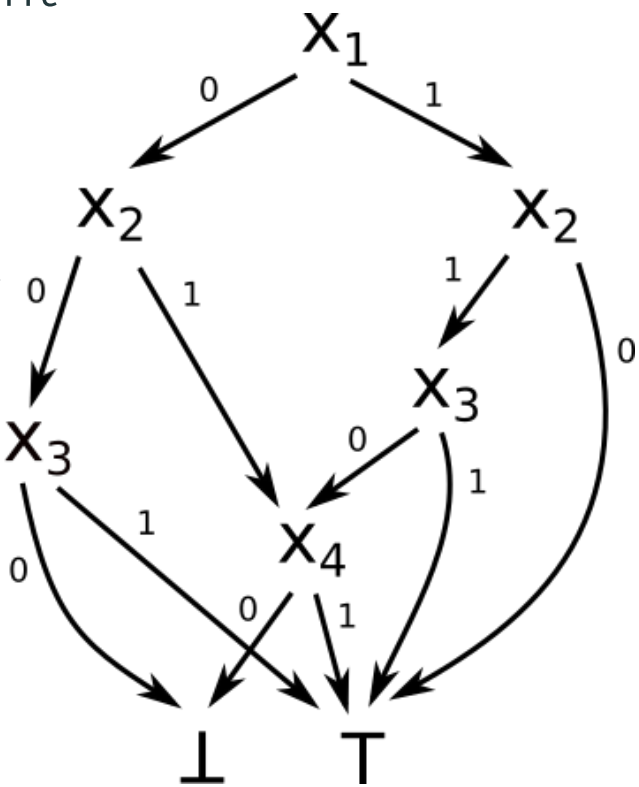

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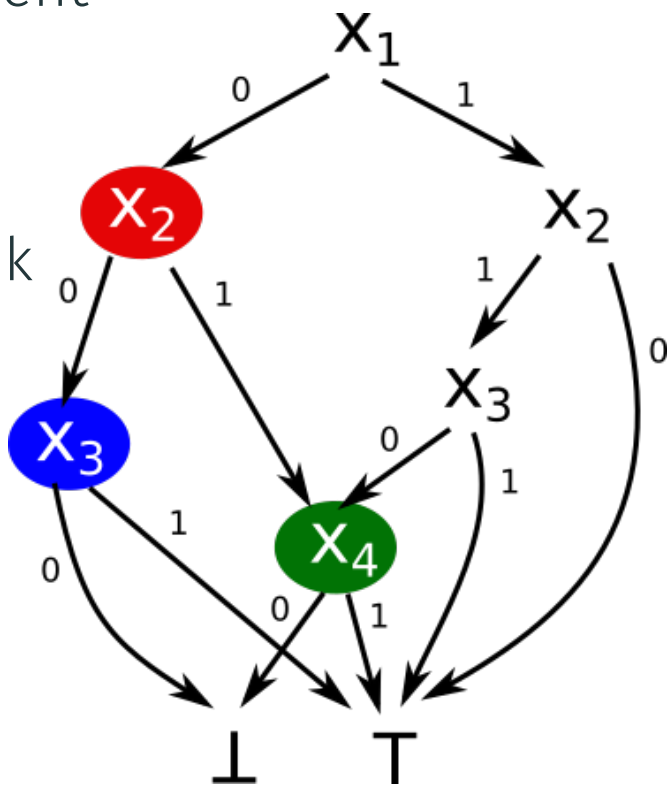

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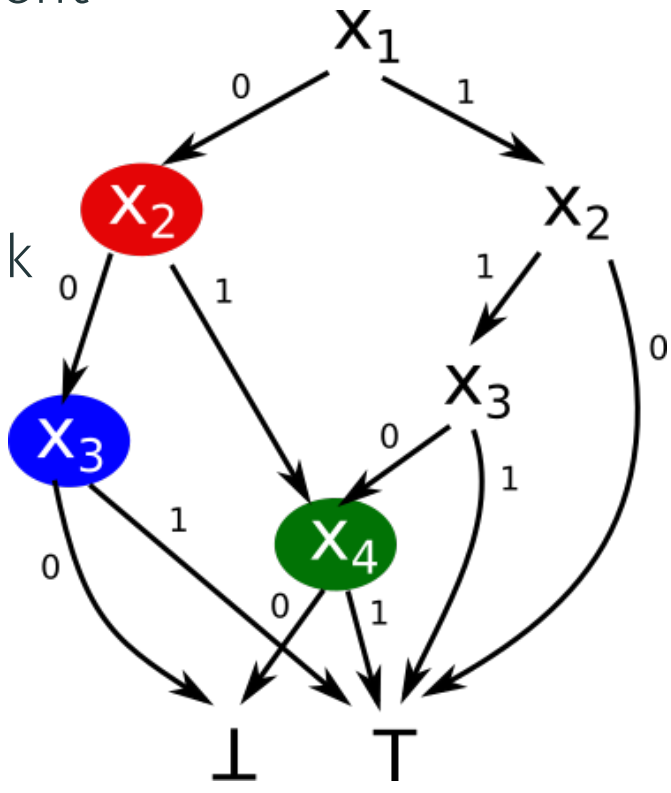

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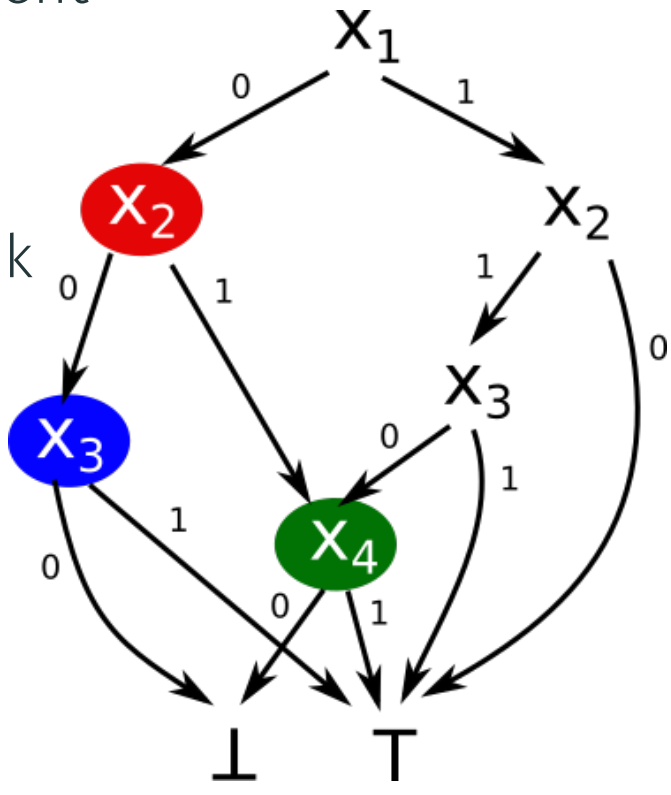

- Width of the OBDD $\simeq$ largest number of nodes that are labeled by the same variable

Pathwidth and OBDDs: Upper and lower bounds

Upper bound:

Theorem (Bova & Slivovsky, 2017)

Let φ be a CNF or DNF of ***pathwidth*** *k. We can compile φ into an* ***OBDD*** *of width 2^{k+2} (hence of size $\leqslant$ nb_vars $\times\, 2^{k+2}$)*

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- Again, this is a generic lower bound!
- For monotone DNF/CNF φ of constant arity and degree, the smallest width of an OBDD for φ is $2^{\Theta(\mathrm{pathwidth}(\varphi))}$

Application to provenance

Provenance: definition

Definition

The *provenance* $\mathsf{Prov}(q, I)$ of query q on relational instance I is the Boolean function with facts of I as variables and such that for any valuation $\nu : I \to \{0, 1\}$, $\mathsf{Prov}(q, I)$ evaluates to $\top$ under ν iff $\{F \in I | \nu(F) = 1\} \models q$

Example: Provenance

$\exists x\, y\, z\ (R(x, y) \wedge S(y, z)) \vee (S(x, y) \wedge R(y, z))$

R	
b	c
c	a
c	d

S	
a	b
d	b

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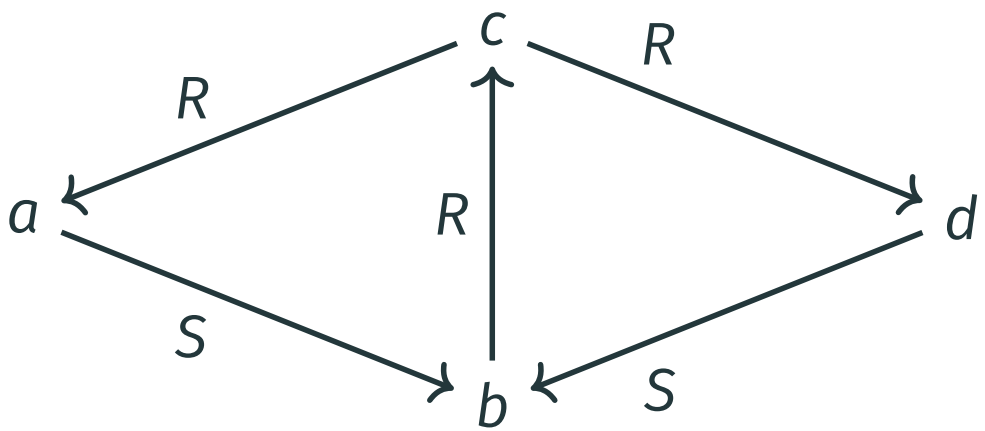

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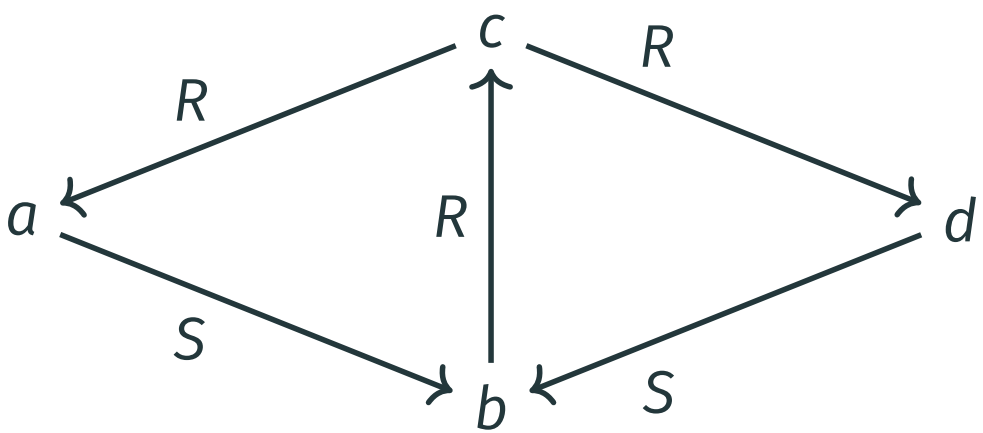

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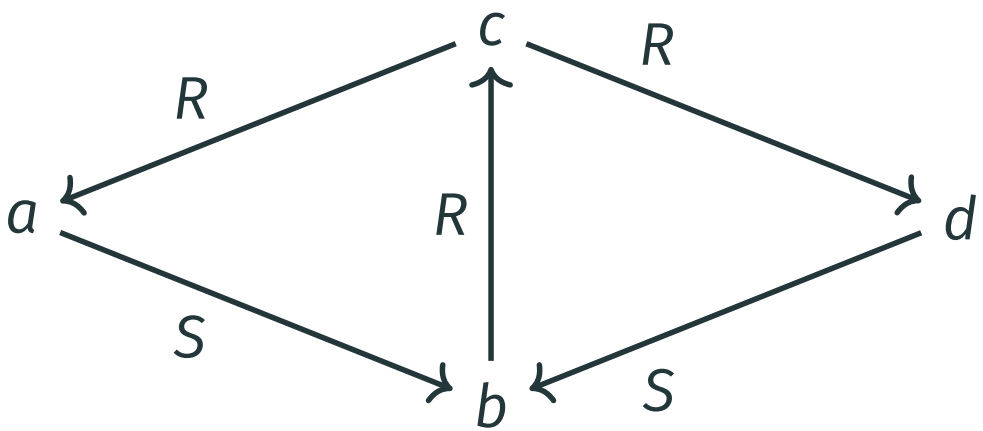

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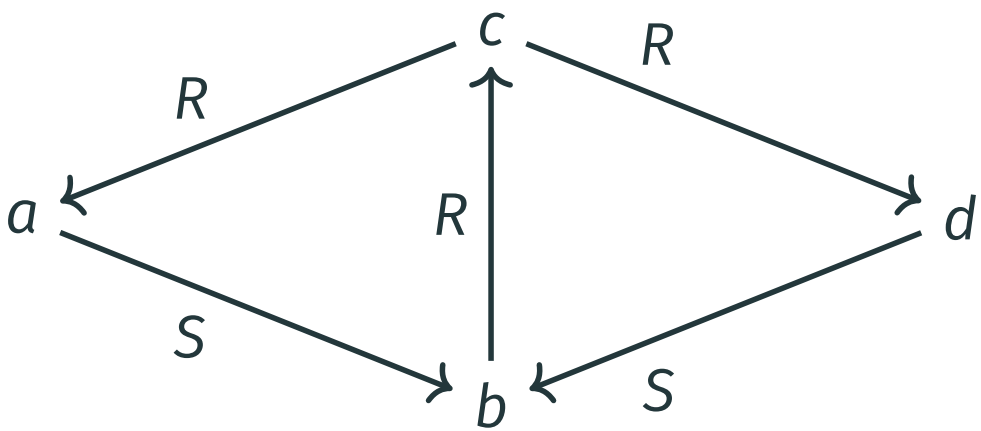

$$\begin{aligned}\mathrm{Prov}(q, I) = &[S(a,b) \wedge (R(b,c) \vee R(c,a))] \\ &\vee [S(d,b) \wedge (R(b,c) \vee R(c,d))]\end{aligned}$$

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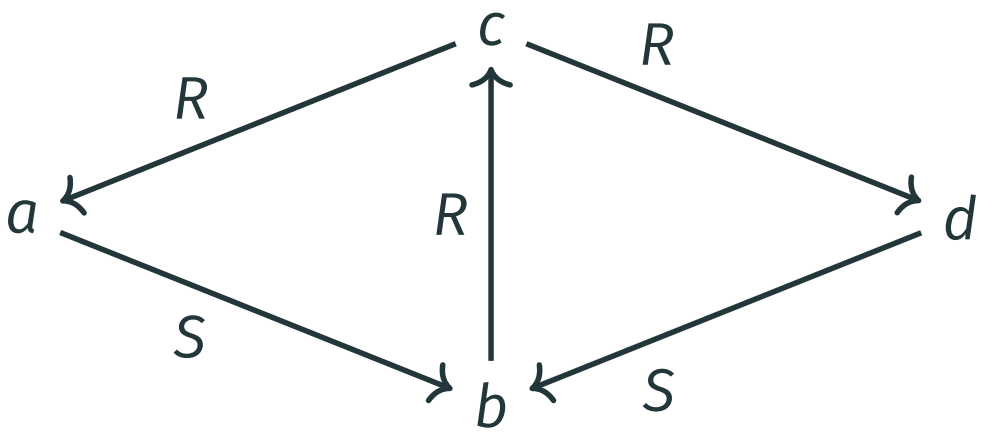

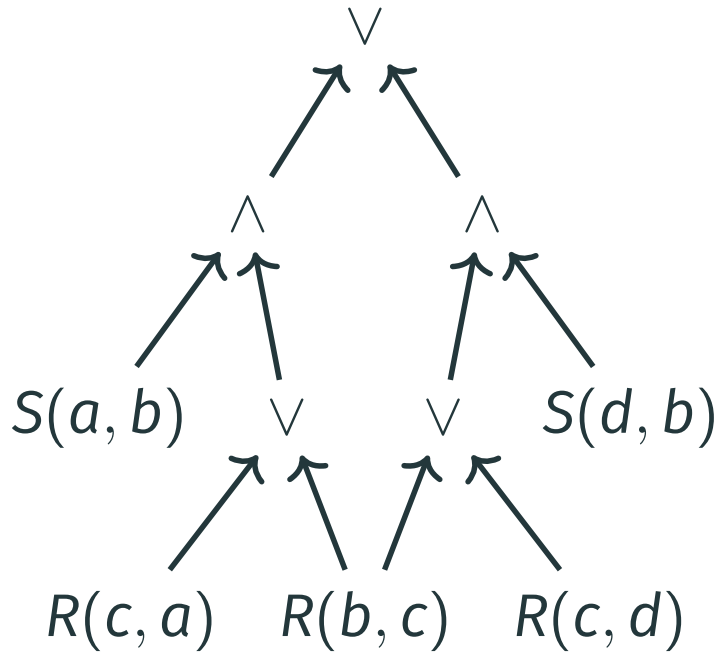

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- Conversely, there are queries for which the lineage as a DNF has same **treewidth** as the instance
- Hence, there are queries for which d-SDNNF representations of the lineage have size exponential in the **treewidth** of the database!

Intricate queries

Theorem (This paper)

*There is a constant $d \in \mathbb{N}$ such that the following is true. Let σ be an arity-2 signature, and Q a connected $\mathrm{UCQ}^{\neq}$ which is **intricate** on σ. For any instance I on σ of **treewidth** k, any **d-SDNNF** representing the lineage of Q on I has size $2^{\Omega(k^{1/d})}$*

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- Future work:
  - → Get rid of arity and degree assumptions?
  - → Notion of width for d-SDNNFs?
  - → Lower bound for d-DNNFs?

Thanks for your attention!