Perfect Matchings in the Powerset

L3 internship topic proposal, Inria Lille, team LINKS

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Abstract: During this internship we propose to work on the following conjecture: if G_k is the underlying undirected graph of the Hasse diagram of the powerset of $V_k = \{0, \dots, k\}$, and U is a set of subsets of V_k that is upward closed and such that $\sum_{s \in U} (-1)^{|s|} = 0$, then either the subgraph of G_k induced by U has a perfect matching, or that induced by $2^{V_k} \setminus U$ has a perfect matching.

Keywords: Perfect matchings, Boolean lattice

1 Topic presentation

Notation. We recall that, if $G = (V, E)$ is an undirected graph, where edges $e \in E$ are of the form $\{a, b\}$ for $a, b \in V$, $a \neq b$, then a *matching* of G is a set $M \subseteq E$ of edges such that for every $e, e' \in M$ we have $e \cap e' = \emptyset$. In other words, no node of G is in two edges of the matching. A matching M is *perfect* if every node of the graph is in some edge of M. For a set $U \subseteq V$ of nodes of G, we define *the subgraph of G induced by U*, denoted $G[U]$, by $G[U] := (U, \{e \in E \mid e \subseteq U\})$. For $k \in \mathbb{N}$, let $V_k := \{0, \dots, k\}$, 2^{V_k} be the powerset of V_k, and G_k be the undirected graph with set of vertices 2^{V_k} and set of edges $\{\{s, s'\} \mid s \subseteq s' \text{ and } |s'| = |s| + 1\}$. In other words G_k is the underlying undirected graph of the Hasse diagram of the powerset of V_k. A set U of nodes of G_k is called *upward closed* if for every $s, s' \in 2^{V_k}$ such that $s \in U$ and $s \subseteq s'$ then we have $s' \in U$.

For instance, considering the graph G_4 from Figure 1, the set U of nodes that are colored in orange is upward closed, the edges that are colored in red in Figure 1 form a perfect matching of $G_k[U]$, and the edges colored in green form a perfect matching of $G_k[2^{V_k} \setminus U]$.

Problem and goal of the internship. We are interested in understanding for which upward closed sets $U \subseteq 2^{V_k}$ of nodes of G_k the graph $G_k[U]$ has a perfect matching. Notice that $G_k[U]$ is a *bipartite graph*, with vertices having even size in one side of the partition and those having odd size in the other partition. For a set $U \subseteq 2^{V_k}$, define the *Euler characteristic of U* to be $\mathsf{Eul}(U) := \sum_{s \in U} (-1)^{|s|}$. Hence, by the preceding observation, for $G_k[U]$ to have a perfect matching it is necessary that $\mathsf{Eul}(U) = 0$. Not all graphs $G_k[U]$ such that U is updward closed and has $\mathsf{Eul}(U) = 0$ have a perfect matching: for instance, considering G_5 from Figure 2, with U again the orange-colored nodes, it can be checked that U is upward-closed, $\mathsf{Eul}(U) = 0$, and yet $G_5[U]$ has no perfect matching. However, in this case we have that $G_5[2^{V_5} \setminus U]$ has a perfect matching (edges in green). This leads to the following conjecture:

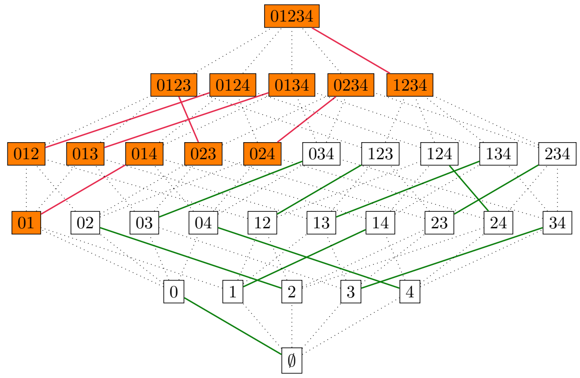

Figure 1: G_4, with an upset of Euler characteristic zero and associated perfect matchings.

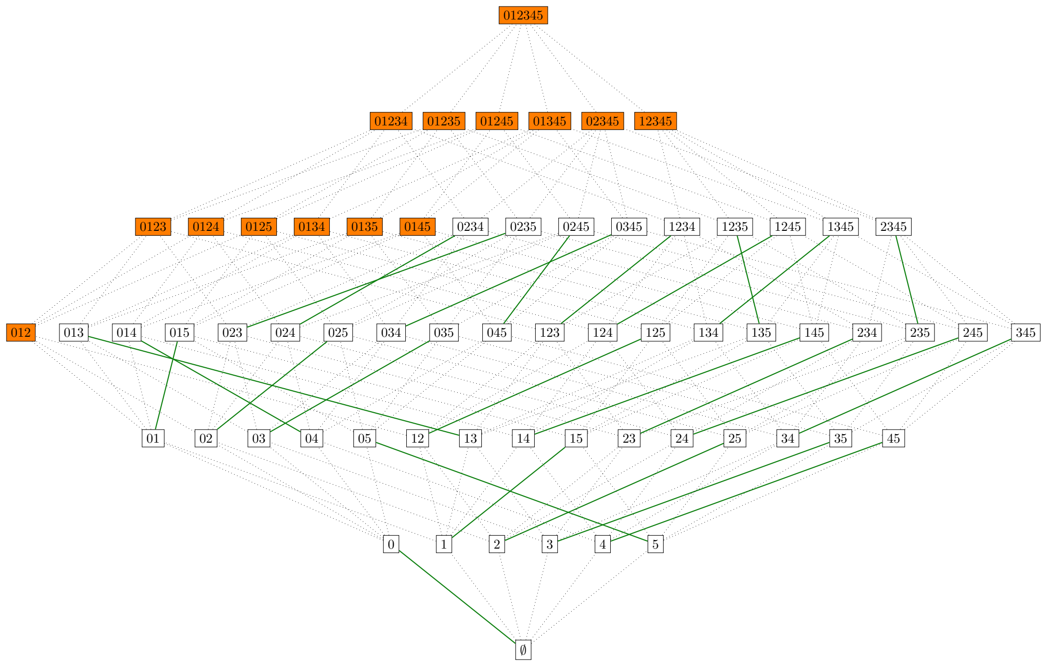

Figure 2: G_5, with an upset of Euler characteristic zero and a perfect matching only for the complement graph.

Conjecture 1 *Let $k \in \mathbb{N}$, and let $U \subseteq 2^{V_k}$ be upward closed with $\mathsf{Eul}(U) = 0$. Then either $G_k[U]$ or $G_k[2^{V_k} \setminus U]$ has a perfect matching.*

The goal of this internship would be to either prove this conjecture to be true, or to find a counterexample to it. Currently, the conjecture has been tested to $k = 0, \dots, 6$ and no counterexample was found. The candidate would work in parallel on proving the conjecture (for instance, proposing generalizations of it that would make it easier to prove) and on disproving it (by brute-force coding, for instance by generating random examples and testing the conjecture on them).

Motivation. The motivation for this problem comes from *probabilistic query evaluation*, but is out of scope of such a short internship. If true, the conjecture would allow to slightly improve a result appearing in [Mon20].

2 Context

The internship will be carried out in LINKS[1], which is a joint research team between Inria Lille[2], the University of Lille[3], and the CRIStAL laboratory[4]. It will be supervised by Mikaël Monet[5], an Inria full-time researcher working on theoretical aspects of uncertain data management, knowledge compilation, and more recently on applying symbolic and logical approaches to explainable AI.

3 How to apply

This proposal is for a short internship (about 6 weeks). We are looking for someone with a certain taste for combinatorial problems. Simply contact us at mikael.monet@inria.fr if you are interested!

References

[Mon20] Mikaël Monet. Solving a special case of the intensional vs extensional conjecture in probabilistic databases. In *PODS*. ACM, 2020.

[1] https://team.inria.fr/links/
[2] https://www.inria.fr/en/centre-inria-lille-nord-europe
[3] https://www.univ-lille.fr/home/
[4] https://www.cristal.univ-lille.fr/en/
[5] https://mikael-monet.net/