Probabilistic Graph Homomorphism

Antoine Amarilli[1], **Mikaël Monet**[1,2], Pierre Senellart[2,3]

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[1]LTCI, Télécom ParisTech, Université Paris-Saclay; Paris, France

[2]Inria Paris; Paris, France

[3]École normale supérieure, PSL Research University; Paris, France

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$$\longrightarrow \quad \longrightarrow$$

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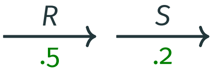

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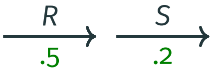

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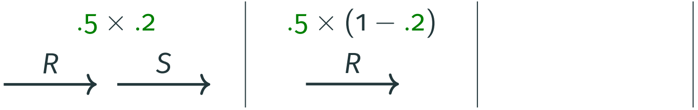

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We write $G \rightsquigarrow H$ if there exists a homomorphism from G to H

Probabilistic Graph Homomorphism (PHom)

Let us fix:

- Finite set of labels Σ
- Class $\mathcal{G}$ of query graphs on Σ (e.g., paths, trees)
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Probabilistic Graph Homomorphism (PHom) problem for $\mathcal{G}$ and $\mathcal{H}$:

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$\rightarrow$ $\Pr(G \rightsquigarrow H) = \sum_{J \subseteq H,\ G \rightsquigarrow J} \Pr(J)$

Example

$$G = \quad x \xrightarrow{R} y \xrightarrow{S} z \xleftarrow{S} t$$

$$H = \quad \bullet \xrightarrow[.2]{R} \bullet \xrightarrow[.5]{S} \bullet \xleftarrow[.8]{R} \bullet$$

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$\Pr(G \rightsquigarrow H) = .2 \times .5$

Complexity of Probabilistic Graph Homomorphism

Question: what is the complexity of PHom
depending on the class $\mathcal{G}$ of query graphs and class $\mathcal{H}$ of instance graphs?

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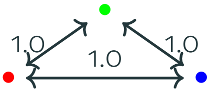
Like CSP but with probabilities!

Fix one side

- Fix the instance graph $H =$  NP-hard

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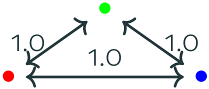
- Fix the instance graph $H =$ 

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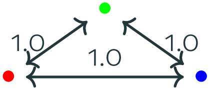
- Fix the instance graph $H =$ 

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To make PHom tractable, we must restrict both sides

Restrict instance graphs to trees

$\mathcal{G} =$ one-way paths (1WP), $\mathcal{H} =$ polytrees (PT)

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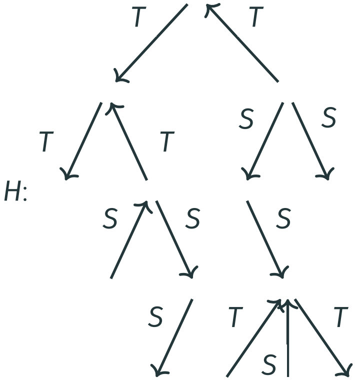

+ prob. for each edge

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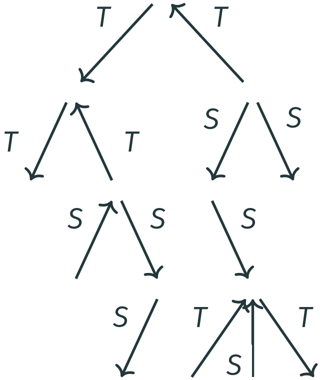
H: 

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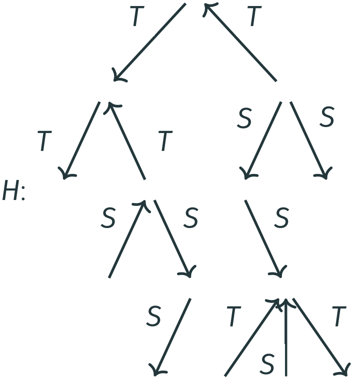
PHom of 1WP on PT is **#P-hard**!

$\mathcal{G}$ = *one-way paths*, $\mathcal{H}$ = *polytrees*, without labels

- What if we do not have labels?



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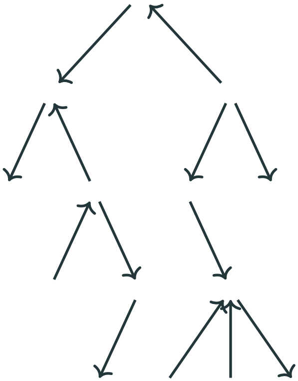

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$\mathcal{G}$ = ***one-way paths***, $\mathcal{H}$ = ***polytrees***, **without labels**

- What if we do not have labels?
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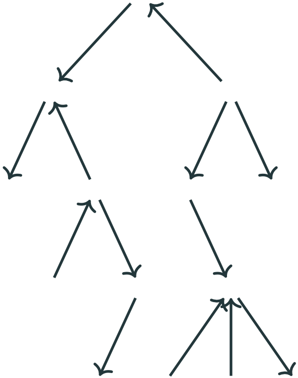
H: 

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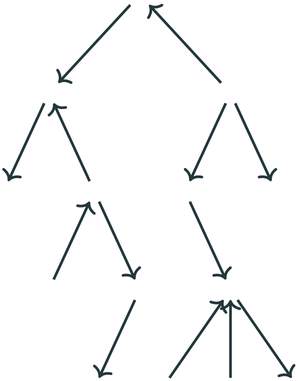

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- Labels have an impact!

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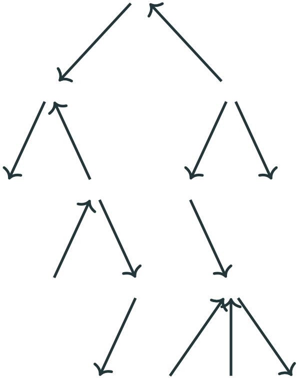

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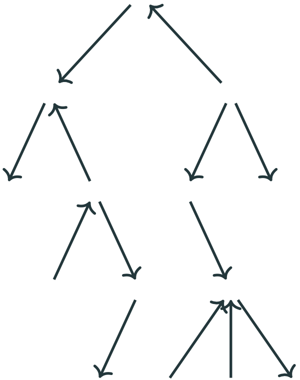
$\mathcal{G}$ = *two-way paths*, $\mathcal{H}$ = *polytrees*, without labels

- $\mathcal{G}$ = one-way paths (1WP), $\mathcal{H}$ = polytrees (PT)

G: 

H: 

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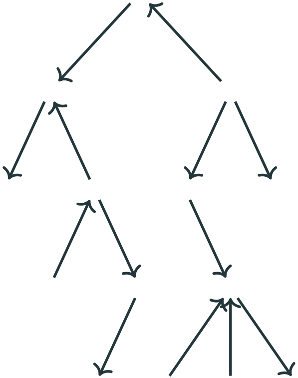

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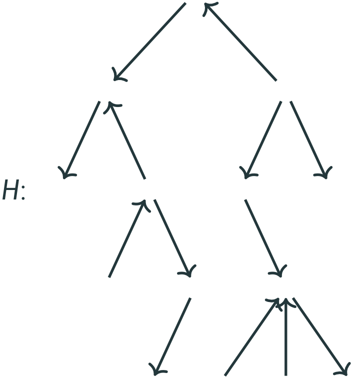

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- Global orientation of the query has an impact

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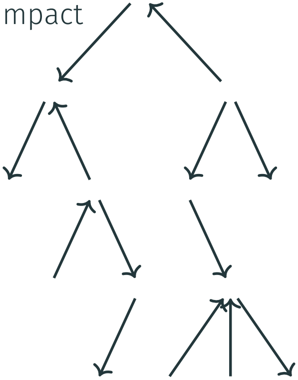

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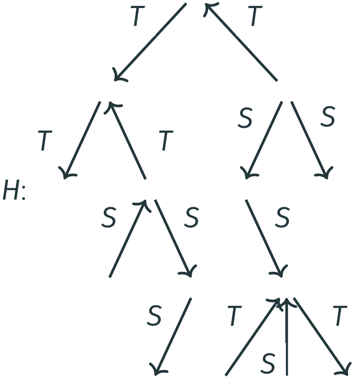

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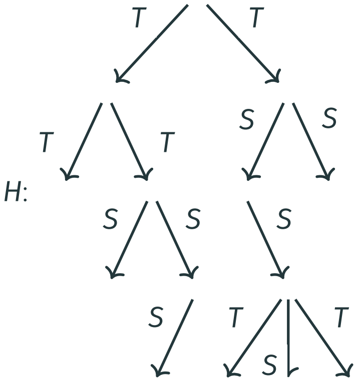

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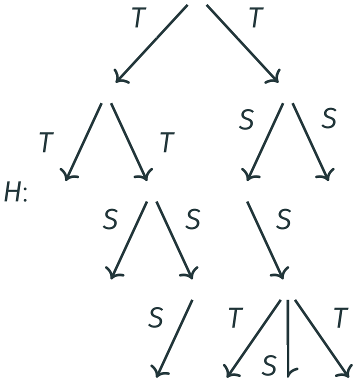

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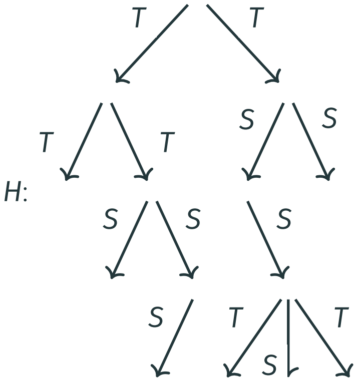
- Global orientation of the instance also has an impact!



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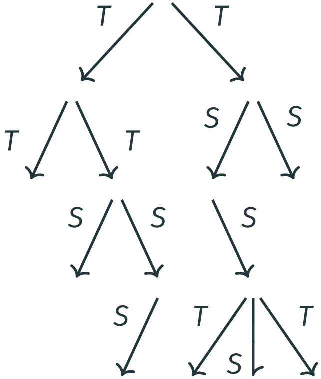
H:



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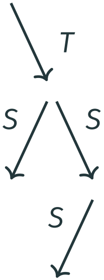

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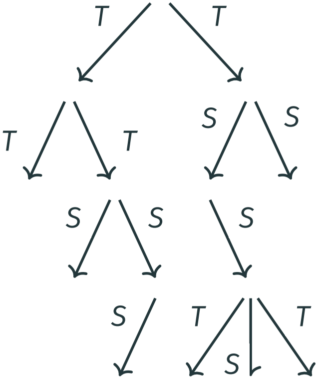

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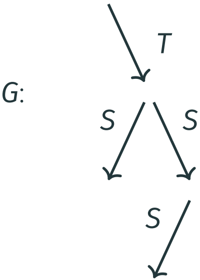

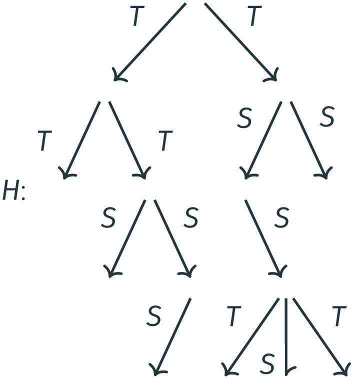

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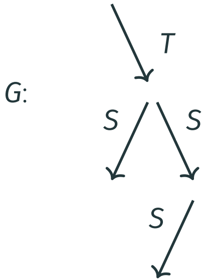
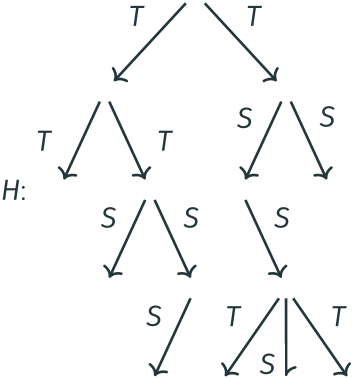
- Branching has an impact!





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Results

$\downarrow G$ $\quad H\rightarrow$	1WP	2WP	DWT	PT	Connected
1WP					
2WP					
DWT		PTIME			
PT					#P-hard
Connected					

$\geqslant 2$ labels

$\downarrow G$ $\quad H\rightarrow$	1WP	2WP	DWT	PT	Connected
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2WP					
DWT		PTIME			
PT					#P-hard
Connected					

No labels

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Thank you for your attention!