Connecting Width and Structure in Knowledge Compilation

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- You have a task
  - → Boolean SAT (is there a satisfying assignment?)

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- You have a task
  - → Boolean SAT (is there a satisfying assignment?)
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  - → enumeration
- Idea: compile the input into a format that is *designed* to solve efficiently your task

Why would I do that?

- Without knowledge compilation

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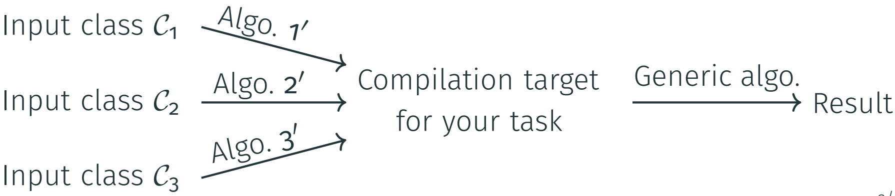
- With knowledge compilation: modularity!



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- We are interested in **#SAT** and **probability evaluation**

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Question: what are the links beetween the two?

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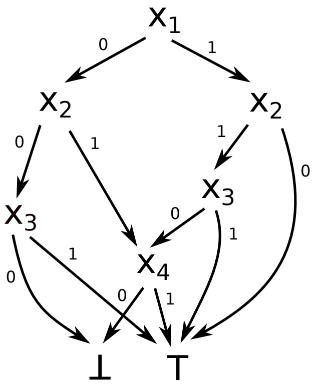
Pathwidth and **OBDDs**

Ordered Binary Decision Diagrams (OBDDs)

- DAG with sink nodes $\{\top, \bot\}$ and internal nodes labeled by variables



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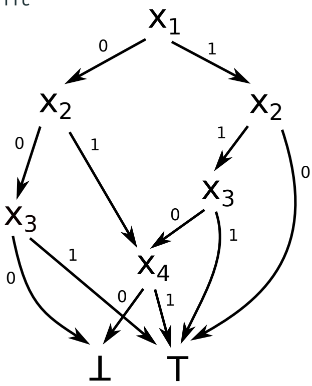

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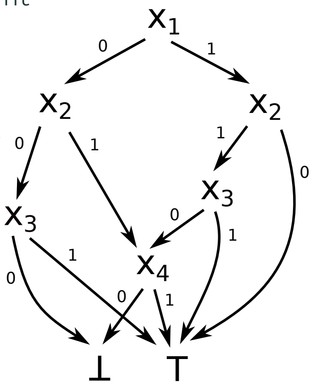

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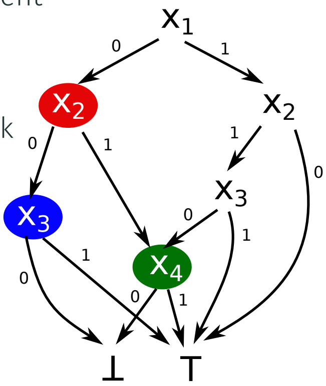

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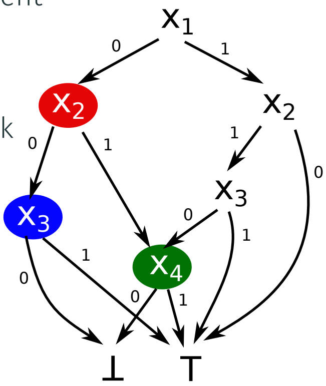

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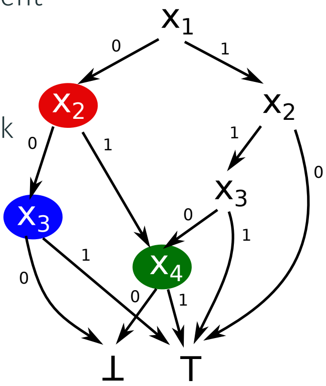
- Width of the OBDD $\simeq$ largest number of nodes that are labeled by the same variable



Bounded pathwidth CNFs/DNFs

- **Pathwidth** of a DNF/CNF: that of its **Gaifman graph**
- Arity: size of the largest clause
- Degree: maximal number of clauses to which a variable belongs

Pathwidth and OBDDs: Upper and lower bounds

Upper bound:

Theorem (Bova & Slivovsky, 2017)

*Let φ be a CNF or DNF of **pathwidth** k. We can compile φ into an **OBDD** of width 2^{k+2} (hence of size $\leqslant nb\_vars \times 2^{k+2}$)*

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Theorem (Our contribution)

*Let φ be a **monotone** CNF or DNF of **pathwidth** k, and let $a := \mathrm{arity}(\varphi)$ and $d := \mathrm{degree}(\varphi)$. Then any **OBDD** for φ has width $\geqslant 2^{\left\lfloor \frac{k}{a^3 \times d^2} \right\rfloor}$*

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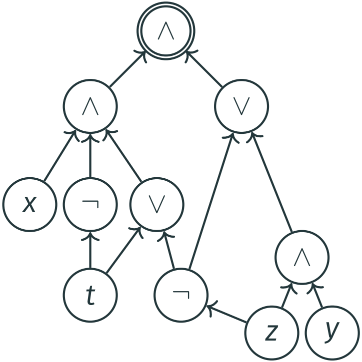
- This is a generic lower bound
- For monotone DNF/CNF φ of constant arity and degree, the smallest width of an OBDD for φ is $2^{\Theta(\text{pathwidth}(\varphi))}$

Treewidth and **d-SDNNFs**

Bounded treewidth Boolean circuits



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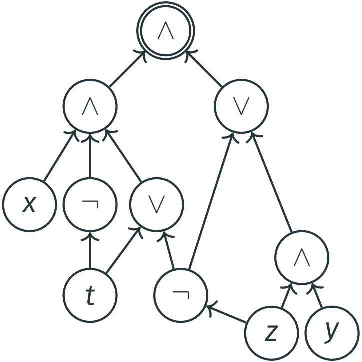

Treewidth of C = that of the underlying graph

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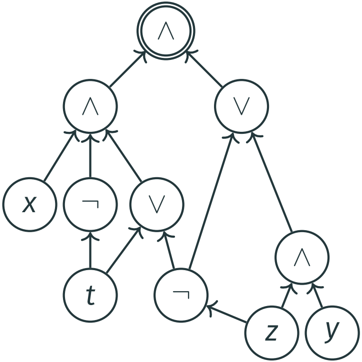

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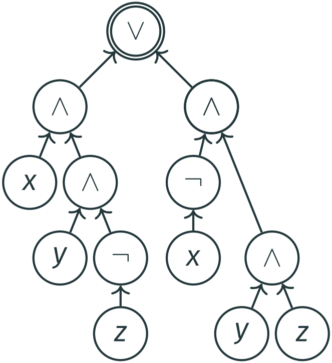
We can do message passing:

Theorem (Lauritzen & Spielgelhalter, 1988)

Fix $k \in \mathbb{N}$. Given a Boolean circuit C of treewidth $\leqslant k$, we can compute its probability in time $O(f(k) \times |C|)$, where f is singly exponential

d-SDNNF



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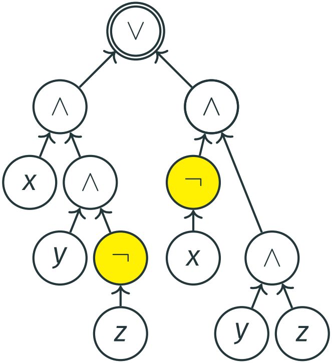

- Negation Normal Form: negations only applied to the leaves

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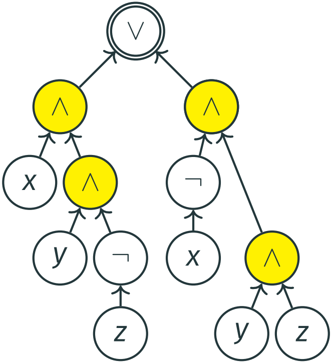

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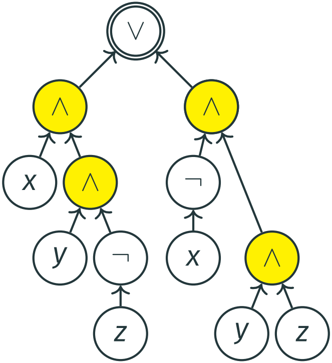

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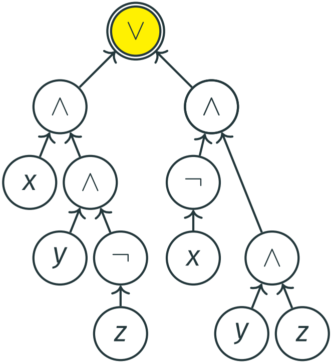

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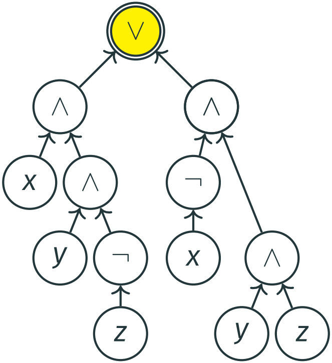

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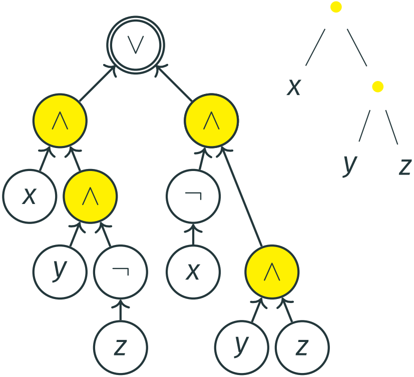
  - → **#SAT** and **probability evaluation**

d-SDNNF



- Negation Normal Form: negations only applied to the leaves
- Decomposable: inputs of ∧-gates are **independent** (no variable has a path to two different inputs of the same ∧-gate)
  - → **SAT** can be solved efficiently
- Deterministic: inputs of ∨-gates are **mutually exclusive**
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- Structured: there is a **vtree** that structures the ∧-gates

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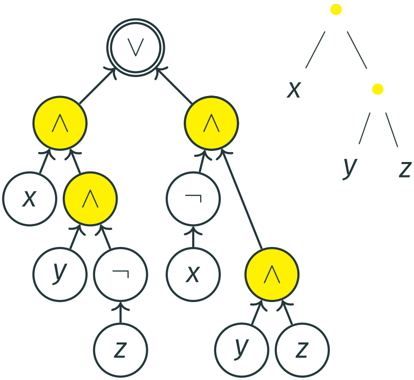

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  - → **Enumeration**

Treewidth and d-SDNNFs: Upper bound

Theorem (Bova & Szeider, 2017)

*Let C be a Boolean circuit on m variables of **treewidth** $\leqslant k$.*
***There exists** a **d-SDNNF** equivalent to C of size $O(m \times g(k))$,*
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Theorem (Our contribution)

Let C be a Boolean circuit of ***treewidth*** $\leqslant k$.
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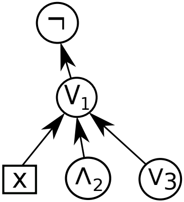
Applications: recapturing message passing, and enumeration of satisfying valuations

Construction sketch



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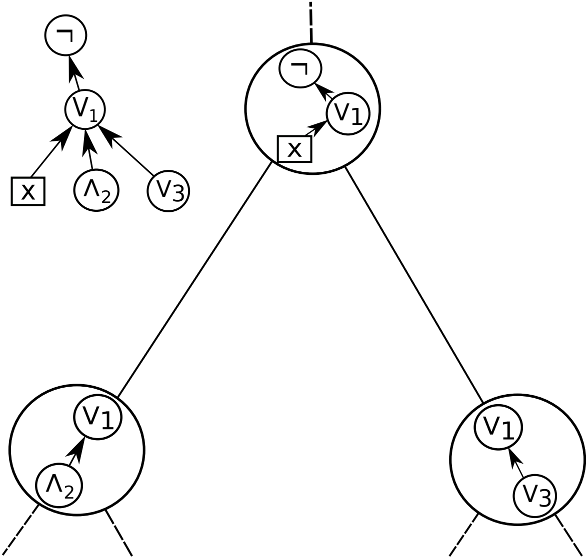

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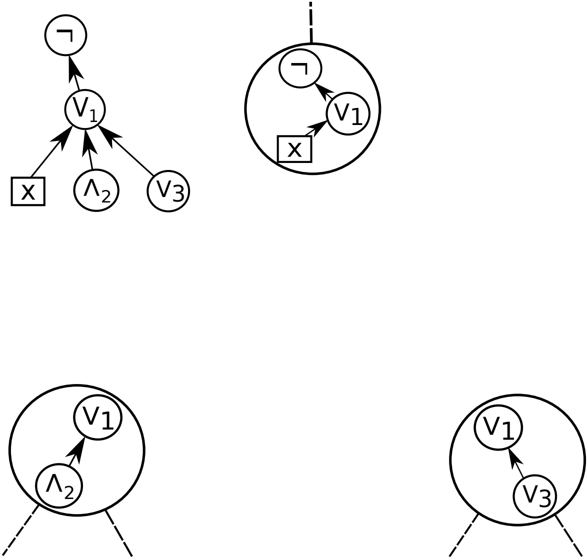

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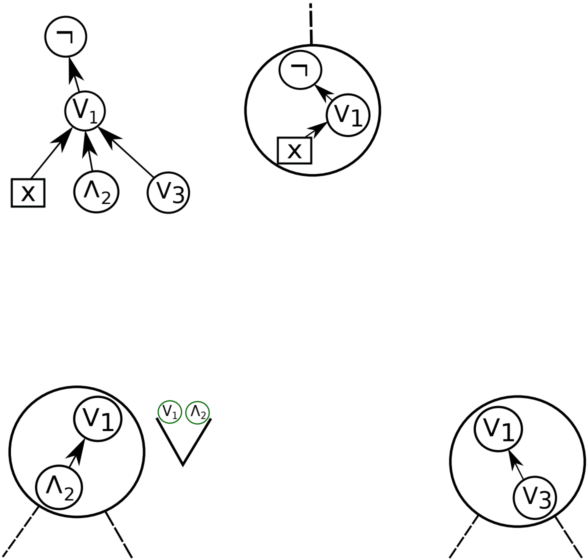

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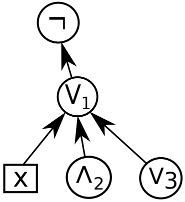

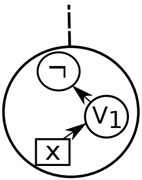

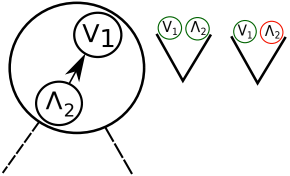

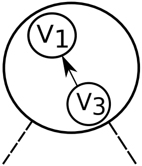

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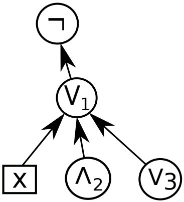

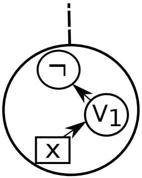

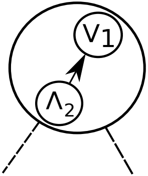

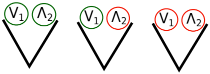

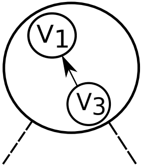

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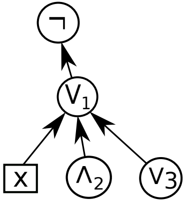

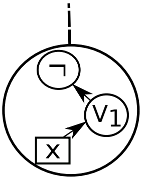

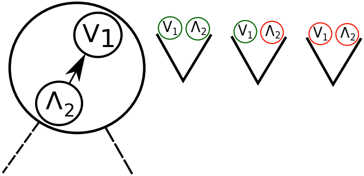

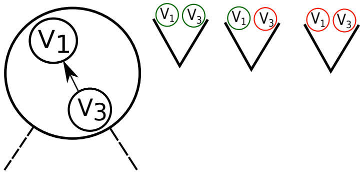

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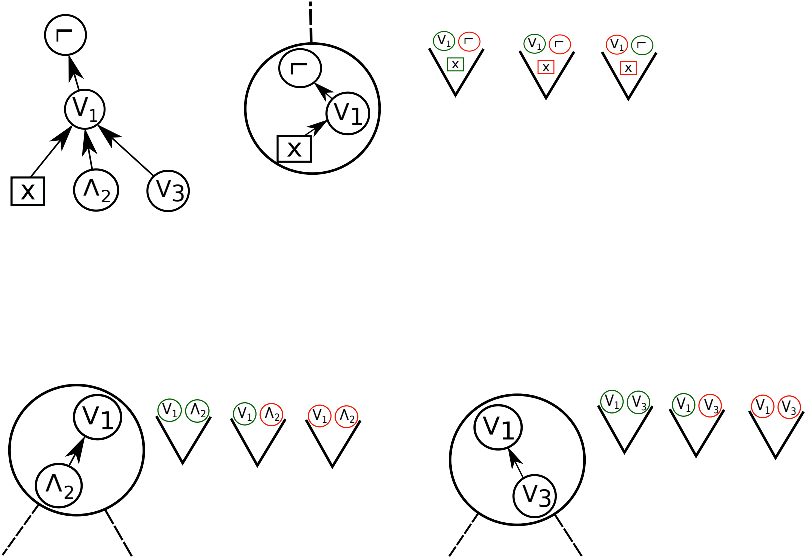

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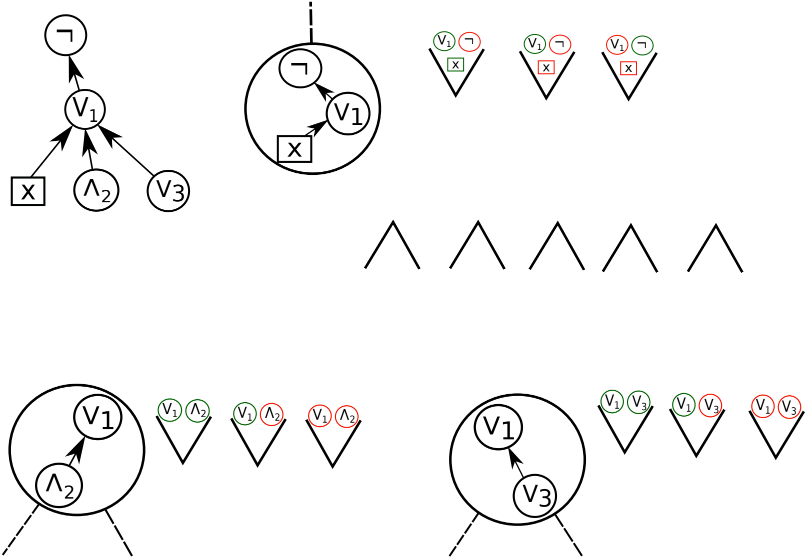

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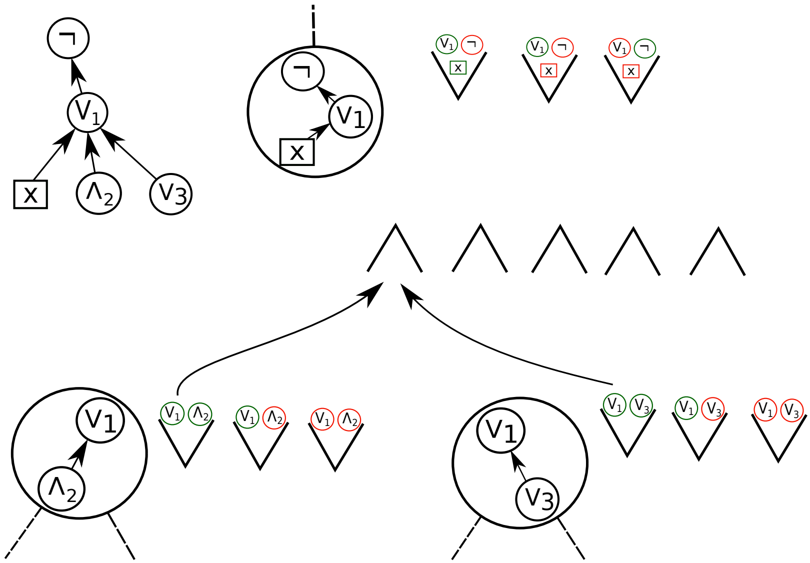

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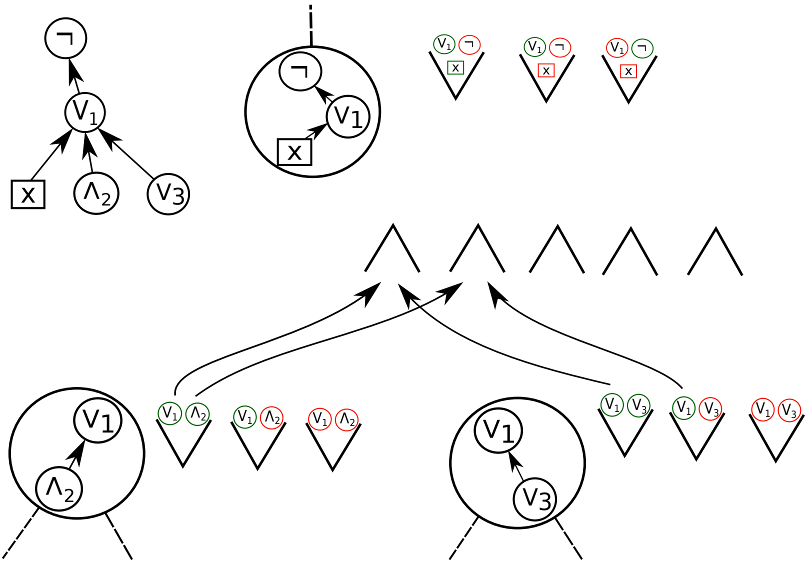

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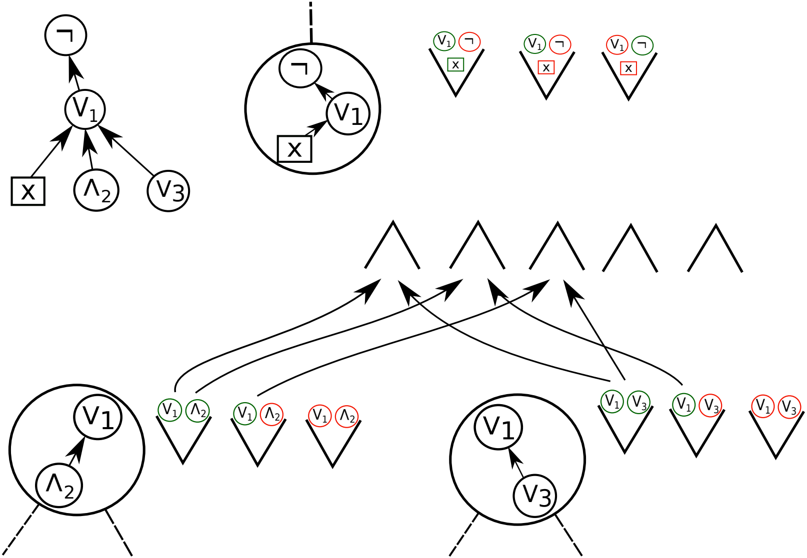

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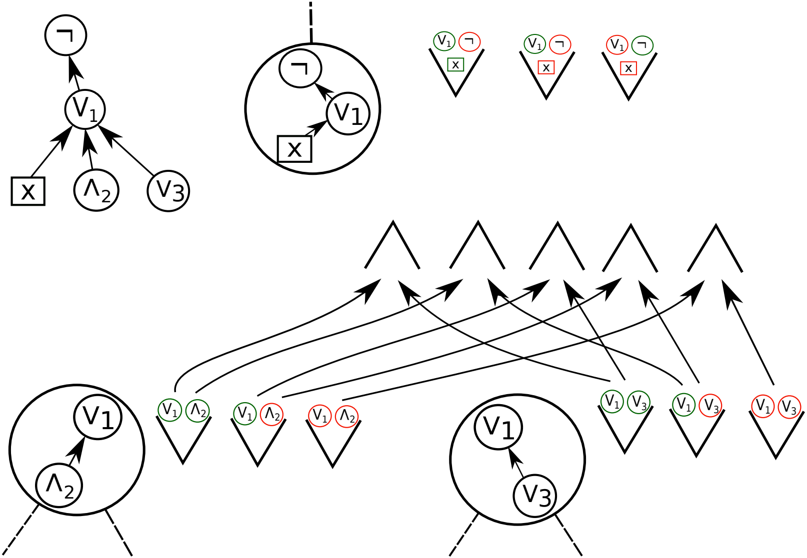

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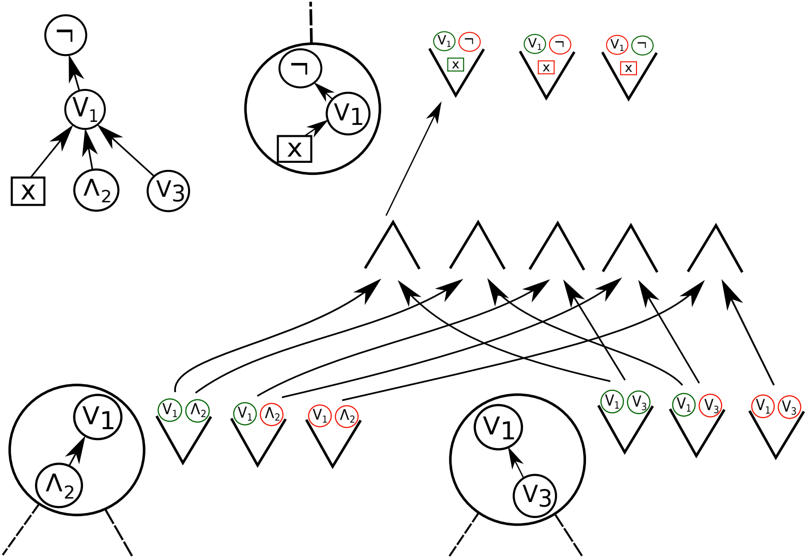

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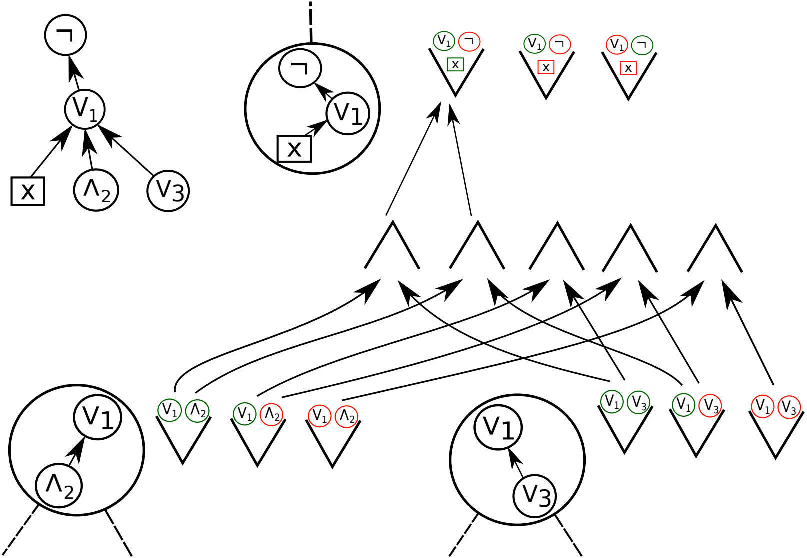

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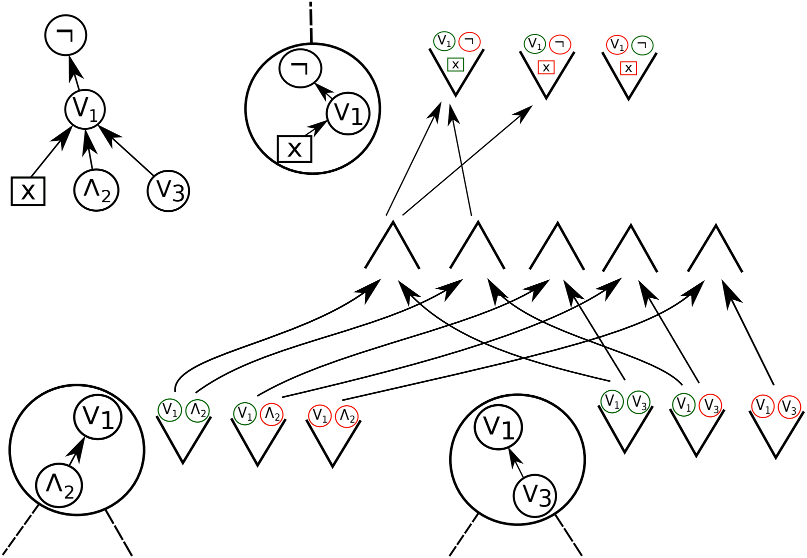

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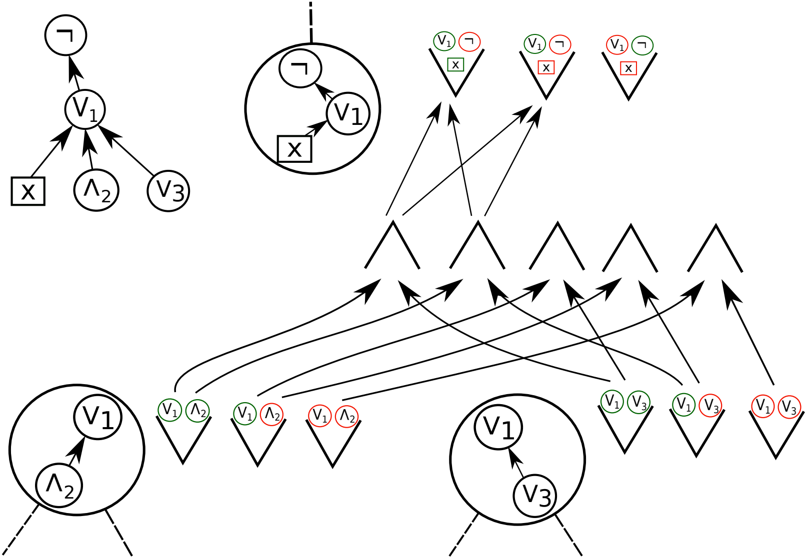

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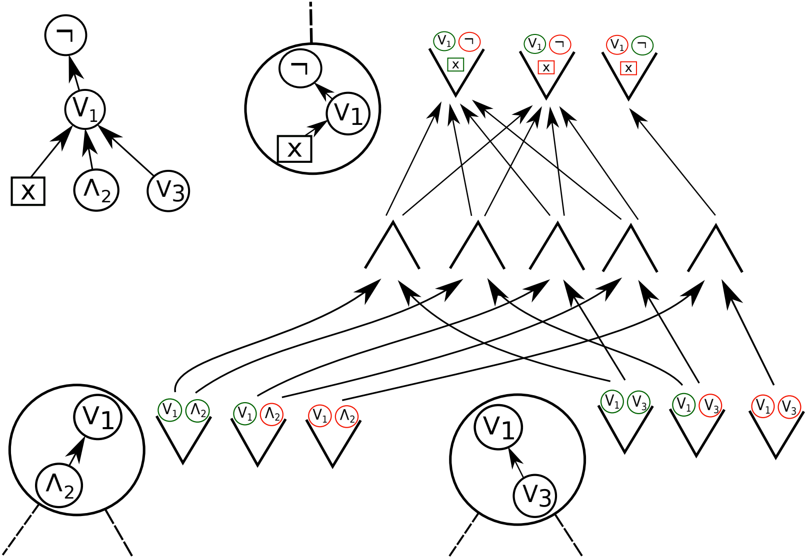

Treewidth and d-SDNNFs: Lower bound

Theorem

*Let φ be a **monotone DNF** of **treewidth** k, let $a := \mathrm{arity}(\varphi)$ and $d := \mathrm{degree}(\varphi)$. Then any **d-SDNNF** for φ has size* $\geqslant 2^{\left\lfloor \frac{k}{3 \times a^3 \times d^2} \right\rfloor} - 1$

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- For **CNFs**, the bound even works for (non-deterministic) **SDNNF**
- Again, the bound is generic: it applies to *any* monotone DNF/CNF

Proof Sketch for CNFs (1/2)

Use the connection made in [Bova, Capelli & Mengel, 2016] between the notion of combinatorial rectangle in **communication complexity** and SDNNFs.

Definition

A (X, Y)-rectangle is a Boolean function $R : 2^{X \cup Y} \to \{0, 1\}$ that can be written as $R_X \land R_Y$, for some Boolean functions $R_X : 2^X \to \{0, 1\}$ and $R_Y : 2^Y \to \{0, 1\}$. A (X, Y)-rectangle cover of a function $f : 2^{X \cup Y} \to \{0, 1\}$ is a set $\{R_1, \cdots, R_n\}$ of (X, Y)-rectangles such that $f \equiv \bigvee_{i=1}^{n} R_i$.

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Theorem (Bova, Capelli & Mengel, 2016)

Let C be an SDNNF computing a function φ on variables V, structured by a v-tree T. Let $n \in T$, and let (X, Y) be the partition of V that n induces. Then φ has a (X, Y)-rectangle cover of size at most $|C|$.

Proof Sketch for CNFs (2/2)

A CNF having no small rectangle cover:

Theorem (Sherstov, 2014)
Let $X = \{x_1, \ldots, x_n\}$ *and* $Y = \{y_1, \ldots, y_n\}$ *be two disjoint sets of variables. Then any* (X, Y)*-rectangle cover of the Boolean function* $\mathrm{SCOV}_n(X, Y) := \bigwedge_{i=1}^{n} x_i \vee y_i$ *has size* $\geqslant 2^n$.

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→ Rephrase treewidth as treesplitwidth, a new measure capturing the 'performance' of a v-tree

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- Future work:
  - → Get rid of arity and degree assumptions?
  - → Notion of width for d-SDNNFs?
  - → Lower bound for d-DNNFs?

Thanks for your attention!